ADMISSIBILITY AND COMMON BELIEF

GEIR B. ASHEIM AND MARTIN DUFWENBERG

ABSTRACT. The concept of 'fully permissible sets' is defined by an algorithm that eliminates strategy subsets. It is characterized as choice sets when there is common certain belief of the event that each player prefer one strategy to another if and only if the former weakly dominates the latter on the set of all opponent strategies or on the union of the choice sets that are deemed possible for the opponent. The concept refines the Dekel-Fudenberg procedure and captures aspects of forward induction. *JEL* Classification Number: C72.

1. INTRODUCTION

Two different, but related, ideas have re-occurred in deductive game-theoretic analysis:

1. A player should prefer one strategy to another if the former weakly dominates the latter. Such admissibility of a player's preferences — which can be referred to as 'caution' since it means that all opponent strategies are taken into account — has been defended by e.g. Luce & Raiffa ([32], Ch. 13) and is implicit in any procedure that starts out by eliminating all weakly dominated strategies.
2. A player should deem any opponent strategy that is a rational choice infinitely more likely (in the sense of Blume, Brandenburger & Dekel [15], Def. 5.1) than any opponent strategy not having this property. This is equivalent to saying that a player should prefer one strategy to another if the former weakly dominates the latter on the set of rational choices for the opponent. Such admissibility of a player's preferences — which will here be referred to as 'full belief of opponent rationality' — is a key ingredient in the analyses of weak dominance by Samuelson [37] and Börgers & Samuelson [19], and is essentially satisfied by procedures, like 'extensive form rationalizabiliy' (EFR, cf. Pearce [35] and Battigalli [9, 10]) and 'iterated elimination of (all) weakly dominated strategies' (IEWDS), that promote forward induction.

The present paper presents an analysis that combines these ideas in the following manner. A player's preferences over his own strategies, which

Date: 29 February 2000.

Acknowledgements: During our collaboration, which Jörgen Weibull was instrumental in initiating, we have received helpful comments from many scholars, including Pierpaolo Battigalli, Adam Brandenburger, Stephen Morris, and Larry Samuelson. Asheim gratefully acknowledges the hospitality of Northwestern University and financial support from the Research Council of Norway.

depend both on his payoff function and on his beliefs about the strategy choice of his opponent, leads to a *choice set* (i.e. a set of maximal strategies). A player's preferences will be said to be *fully admissibly consistent* with the preferences of his opponent if one strategy is preferred to another if and only if the former weakly dominates the latter

- on the set of all opponent strategies (i.e. 'caution'), or
- on the union of the choice sets that are deemed possible for the opponent (i.e. 'full belief of opponent rationality').

A subset of strategies is a *fully permissible set* if and only if it can be a choice set when there is common certain belief of full admissible consistency, where an event is 'certainly believed' if the complement is Savage-null (cf. subsect. 3.3). Hence, the analysis yields a solution concept that determines a collection of strategy subsets – a family of choice sets – for each player.

1.1. **An Illustration.** We will use G_1 of Fig. 1 to illustrate the consequences of imposing 'caution' and 'full belief of opponent rationality'. The reader may want to study the example and decide what is a "reasonable" outcome before reading on. Since 'caution' means that each player takes all opponent strategies into account, it follows that player 1's preferences over his strategies will be $U \sim M \succ D$ (where $\sim$ and $\succ$ denote indifference and preference, respectively). Player 1 must prefer each of the strategies U and M to the strategy D, because the former strategies weakly dominate D. Hence, U and M are maximal, implying that 1's choice set is $\{U, M\}$.

The requirement of 'full belief in opponent rationality' comes into effect when considering the preferences of player 2. Suppose that 2 is certain that 1 is cautious. What will 2's preferences over her strategies be? Since 2 (as indicated above) should figure out that $\{U, M\}$ is 1's choice set, 2 should deem *each* element of $\{U, M\}$ infinitely more likely than D. This is captured by our assumption that 2 has full belief of 1's rationality. It amounts to the requirement that 2's preferences should respect weak dominance on 1's choice set $\{U, M\}$, regardless of what happens if 1 chooses D. Hence, 2's preferences over her strategies will be $L \succ R$.

Summing up, common certain belief of full admissible consistency leads to the following solution for G_1:

$$\text{1's preferences: } U \sim M \succ D$$
$$\text{2's preferences: } L \succ R$$

This means that $\{U, M\}$ is the unique fully permissible set for player 1 and $\{L\}$ is the unique fully permissible set for player 2.

1.2. **Related Concepts.** Several solution concepts with natural epistemic foundations fail to match this prediction. In the case of rationalizability (Bernheim [14], Pearce [35]) — which in 2-player games corresponds to iterated elimination of strongly dominated strategies — this is perhaps not so surprising. Rationalizability can be understood as a consequence of 'common belief of rational choice' without imposing caution (see e.g. Tan & Werlang

	L	R
U	1, 1	1, 1
M	1, 1	1, 0
D	1, 0	0, 1

FIGURE 1. An illustration (G_1).

[39]), so there is no guarantee that a player always prefers one strategy to another if the former weakly dominates the latter. As an illustration, the strategy D in G_1 can be rationalized by 1 believing that 2 chooses L (which in turn can be rationalized by 2 believing that 1 chooses U or M, etc.). This contradicts that 1 is cautious and chooses a strategy in his choice set.

It is perhaps more surprising that the concept of 'permissibility' does not match our solution of G_1. Permissibility can be given rigorous epistemic foundations in models where the players are cautious (cf. Börgers [18], and especially Brandenburger [20] who coined the term 'permissible'; see also Ben-Porath [12] and Gul [30]). In these models players take into account all opponent strategies, while assigning more weight to a *subset* of those opponent strategies they deem to be rational choices for the opponent. Permissibility implies that only strategies that survive the so-called DF procedure (after Dekel & Fudenberg [24]) — one round of elimination of (all) weakly dominated strategies followed by iterated elimination of strongly dominated strategies — can be chosen. In G_1, this means that 1 cannot choose his weakly dominated strategy D. However, in contrast to our solution where 2 prefers L to R, permissibility allows that 2 chooses R. To exemplify using Brandenburger's [20] approach, this will be the case if 2 deems U to be infinitely more likely than D which in turn is deemed infinitely more likely than M. The problem is that our requirement of 'full belief of opponent rationality' is not satisfied: Player 2 deems D more likely than M even though M is in 1's choice set, while D is not. In Sect. 2 we establish as a general result (Prop. 2) that the concept of fully permissible sets refines the DF procedure.

The procedure of IEWDS yields the same conclusion in G_1 as do the concept of fully permissible sets. However, the concept of fully permissible sets neither refines nor is refined by the procedure of IEWDS (see G_3 in Sect. 2 for a proof by example).

We shall use two games — 'Battle-of-the-sexes-with-an-outside-option' (G_2) and 'Burning money' (G_4) — to illustrate how the concept of fully permissible sets captures a notion of forward induction, although by an argument that differs from the one that usually applies. The usual forward induction argument is illustrated in these games by the procedure of IEWDS and the concept of EFR. The latter concept has recently been given an epistemic foundation by Battigalli & Siniscalchi [11]. In Sect. 6 we will explain how our epistemic analysis differs from theirs and compare our work to other related literature.

1.3. **Organization of the Paper.** Section 2 formally defines the concept of fully permissible sets through an algorithm that eliminates strategy sets that cannot be choice sets under full admissible consistency. General existence as well as other properties are shown. Section 3 introduces epistemic operators, which are used in Sect. 4 to establish an epistemic foundation for the concept of fully permissible sets through the requirement of common certain belief of full admissible consistency. Section 5 contains further examples, while Sect. 6 concludes. Some technical material (including the proofs) are contained in two appendices. For ease of presentation, the analysis will be limited to 2-player games. This is mostly a matter of convenience as everything can be generalized to n-player games (with $n > 2$).

2. An Algorithm

We present in this section an algorithm — 'iterated elimination of choice sets under full admissible consistency' (IECFA) — leading to the concept of 'fully permissible sets'. This concept will in turn be given an epistemic characterization in Sect. 4 by imposing common certain belief of full admissible consistency. We present the algorithm before the epistemic characterization for different reasons:

- IECFA is fairly accessible. By defining it early, we can apply it early, and so offer early indications of the nature of the solution concept that we wish to promote.
- To define IECFA is to point to a parallel between our approach and the concepts of rationalizable strategies and permissible strategies. Even though they are motivated by epistemic assumptions, both concepts turn out to be identical in 2-player games to the set of strategies surviving simple algorithms: respectively, iterated elimination of strongly dominated strategies (IESDS) and the DF procedure.
- Just like IESDS and the DF procedure, IECFA is much easier to use than the corresponding epistemic characterizations. The algorithm should be a handy tool for applied economists who may wish to see it presented early and then spend less time on the details of Sect. 4.

IESDS and the DF procedure iteratively eliminate dominated strategies. In the corresponding epistemic models, these strategies in turn cannot be rational choices, cannot be rational choices given that other players do not use strategies that cannot be rational choices, etc. IECFA is also an elimination procedure. However, the interpretation of the basic item thrown out is not that of a strategy that cannot be a rational choice, but rather that of *a set* of strategies that cannot be the set of maximal strategies (i.e., *a choice set*; cf. subsects. 2.2 and 3.5) for any preferences that are in a given sense consistent with the preferences of the opponent. The specific kind of consistency involved in IECFA — which will be defined in Sect. 4.2

and referred to as 'full admissible consistency' — requires that a player's preferences are characterized by the properties of 'caution' and 'full belief of opponent rationality'. Thus, IECFA does not start with each player's strategy set and then iteratively eliminates strategies. Rather, IECFA starts with each player's collection of non-empty subsets of his strategy set and then iteratively eliminates subsets that cannot be choice sets when the players' preferences satisfy the requirement of 'full admissible consistency'.

2.1. **A Strategic Game.** With $N = \{1, 2\}$ as the set of *players*, let, for each i, S_i denote player i's finite set of *pure strategies* and $u_i : S \to \mathbb{R}$ be a vNM utility function that assigns payoff to any strategy vector, where $S = S_1 \times S_2$ is the set of strategy vectors. Then $G = (S_i, u_i)_{i \in N}$ is a finite *strategic* two-player *game*. Write p_i, r_i, and s_i ($\in S_i$) for pure strategies and x_i and y_i ($\in \Delta(S_i)$) for mixed strategies. Since u_i is a vNM utility function, we may extend u_i to mixed strategies: $u_i(x_i, s_j) = \sum_{s_i \in S_i} x_i(s_i) u_i(s_i, s_j)$.

2.2. **Definition.** Say that x_i *weakly dominates* y_i *on* Q_j ($\subseteq S_j$) if, $\forall s_j \in Q_j$, $u_i(x_i, s_j) \geq u_i(y_i, s_j)$, with strict inequality for some $s_j \in Q_j$. Say that player i's preferences over his own strategies are *admissible on* Q_j ($\neq \emptyset$) if x_i is preferred to y_i whenever x_i weakly dominates y_i on Q_j. Player i's *choice set* is the set of pure strategies that are maximal w.r.t. i's preferences over his own strategies: s_i ($\in S_i$) is in i's choice set if and only if there is no x_i ($\in \Delta(S_i)$) such that x_i is preferred to s_i. As indicated in subsect. 3.5, i's choice set is non-empty and supports any maximal mixed strategy.

In order to describe IECFA and thus define the concept of 'fully permissible sets', let the set Q_j be interpreted as the set of strategies that player i deems to be the set of rational choices for his opponent. Assume that player i's preferences over his own strategies are characterized by the property of being admissible on both Q_j and S_j: x_i is preferred to y_i if and only if x_i weakly dominates y_i on Q_j or S_j. Player i's choice set is then equal to $S_i \backslash D_i(Q_j)$, where, for any ($\emptyset \neq$) $Q_j \subseteq S_j$,

$$D_i(Q_j) := \{s_i \in S_i | \exists x_i \in \Delta(S_i) \text{ s.t. } x_i \text{ weakly dom. } s_i \text{ on } Q_j \text{ or } S_j\} \, .$$

Let $\Sigma = \Sigma_1 \times \Sigma_2$, where $\Sigma_i := 2^{S_i} \backslash \{\emptyset\}$ denotes the collection of non-empty subsets of S_i. Write π_i, ρ_i, and σ_i ($\in \Sigma_i$) for subsets of pure strategies. For any ($\emptyset \neq$) $\Xi = \Xi_1 \times \Xi_2 \subseteq \Sigma$, write $\alpha(\Xi) := \alpha_1(\Xi_2) \times \alpha_2(\Xi_1)$, where

$$\alpha_i(\Xi_j) := \{\pi_i \in \Sigma_i | \exists (\emptyset \neq) \Psi_j \subseteq \Xi_j \text{ s.t. } \pi_i = S_i \backslash D_i(\cup_{\sigma_j \in \Psi_j} \sigma_j)\} \, .$$

Hence, $\alpha_i(\Xi_j)$ is the collection of strategy subsets that can be choice sets for player i if i's preferences are characterized by the property of being admissible *both* on the union of the strategy subsets in a non-empty subcollection of Ξ_j *and* on the union of all opponent strategies.

We can now define the main concept of this paper.

Definition 1. Consider the sequence defined by $\Xi(0) = \Sigma$ and, $\forall g \geq 1$, $\Xi(g) = \alpha(\Xi(g-1))$. A non-empty strategy set π_i is said to be a *fully permissible set* for i if $\pi_i \in \bigcap_{g=0}^{\infty} \Xi_i(g)$.

	L	R
U	2, 2	2, 2
M	3, 1	0, 0
D	0, 0	1, 3

FIGURE 2. Battle-of-the-sexes-with-an-outside-option (G_2).

Let $\Pi = \Pi_1 \times \Pi_2$ denote the *collection* of vectors of fully permissible sets. Since $\emptyset \neq \alpha_i(\Xi'_j) \subseteq \alpha_i(\Xi''_j) \subseteq \alpha_i(\Sigma_j)$ whenever $\emptyset \neq \Xi'_j \subseteq \Xi''_j \subseteq \Sigma_j$ and since the game is finite, $\Xi(g)$ is a monotone sequence that converges to Π in a finite number of iterations. IECFA is the procedure that in round g eliminates sets in $\Xi(g-1)\backslash\Xi(g)$ as possible choice sets. As defined in Def. 1 IECFA eliminates maximally in each round in the sense that, $\forall g \geq 1$, $\Xi(g) = \alpha(\Xi(g-1))$. However, it follows from the monotonicity of α_i that any non-maximal procedure, where $\exists g \geq 1$ such that $\Xi(g-1) \supset \Xi(g) \supset \alpha(\Xi(g-1))$, will also converge to Π.

A choice set of player i survives elimination round g if it is a choice set w.r.t. preferences that are characterized by the property of being admissible *both* on the union of some (or all) of opponent choice sets that have survived the procedure up till round $g-1$ *and* on the set of all opponent strategies. A fully permissible set is a choice set which will survive in this way for any g. It will follow from the analysis of Sect. 4 that strategy subsets that this algorithm has not eliminated by round g can be interpreted as choice sets that are compatible with $g-1$ order of mutual certain belief of full admissible consistency.

2.3. **Applications.** The best way to illustrate the functioning of IECFA is to apply it. Consider first G_1 of the introduction. We get:

$$\begin{aligned}\Xi(0) &= \Sigma_1 \times \Sigma_2 \\ \Xi(1) &= \{\{U,M\}\} \times \Sigma_2 \\ \Pi = \Xi(2) &= \{\{U,M\}\} \times \{\{L\}\}\,.\end{aligned}$$

Independently of Q_2, $S_1\backslash D_1(Q_2) = \{U,M\}$, so for 1 only $\{U,M\}$ can survive the first elimination round. On the other hand, $S_2\backslash D_2(\{U,M\}) = \{L\}$, $S_2\backslash D_2(\{D\}) = \{R\}$, and $S_2\backslash D_2(\{U\}) = \{L,R\}$, so that no elimination is possible for player 2. However, in the second elimination round only $\{L\}$ survives since $S_2\backslash D_2(\{U,M\}) = \{L\}$. The interpretation is that (in round 2) it is impossible for R to appear in a choice set for 2. This is because (in round 1) only $\{U,M\}$ is possible as a choice set for 1, and then $\{L\}$ must be 2's choice set since only L is a maximal element w.r.t. preferences are admissible on $\{U,M\}$ and $\{U,M,D\}$.

We now consider a different example, the 'Battle-of-the-sexes-with-an-outside-option' game, which pure strategy reduced strategic form is given by G_2 of Fig. 2. The example shows that our solution concept captures a notion of forward induction. We will return to this example on several occasions throughout the paper. Applying IECFA we get:

$$\begin{aligned}
\Xi(0) &= \Sigma_1 \times \Sigma_2 \\
\Xi(1) &= \{\{U\},\{M\},\{U,M\}\} \times \Sigma_2 \\
\Xi(2) &= \{\{U\},\{M\},\{U,M\}\} \times \{\{L\},\{L,R\}\} \\
\Xi(3) &= \{\{M\},\{U,M\}\} \times \{\{L\},\{L,R\}\} \\
\Xi(4) &= \{\{M\},\{U,M\}\} \times \{\{L\}\} \\
\Pi = \Xi(5) &= \{\{M\}\} \times \{\{L\}\}
\end{aligned}$$

We move directly to the interpretation in terms of surviving choice sets: *(Round 1)* D cannot be in a choice set for 1 since this strategy is strictly dominated. *(Round 2)* This implies that $\{R\}$ is excluded as a choice set for 2, since only $\{U\}$, $\{M\}$, and $\{U,M\}$ are candidates for 1's choice set and $S_2 \backslash D_2(Q_1) \neq \{R\}$ if Q_1 is the union of some (or all) of the sets $\{U\}$, $\{M\}$, and $\{U,M\}$. *(Round 3)* Similarly, $\{U\}$ is excluded as a choice set for 1, since only $\{L\}$ and $\{L,R\}$ are candidates for 2's choice set. *(Round 4)* Given this $\{L,R\}$ cannot be 2's choice set. *(Round 5)* Since $\{M\}$ is the set of 1's strategies that are maximal w.r.t. preferences that are admissible on $\{L\}$, only $\{M\}$ survives as a choice set for player 1. Now the algorithm comes to a stop: $S_2 \backslash D_2(\{M\}) = \{L\}$ and $S_1 \backslash D_1(\{L\}) = \{M\}$, and hence $\{M\}$ and $\{L\}$ are the fully permissible sets.

2.4. **Results.** The following proposition characterizes the strategy subsets that survive IECFA and thus are fully permissible.

Proposition 1. *(i)* $\forall i \in N$, $\Pi_i \neq \emptyset$. *(ii)* $\Pi = \alpha(\Pi)$. *(iii)* $\forall i \in N$, $\pi_i \in \Pi_i$ *if and only if there exists* $\Xi = \Xi_1 \times \Xi_2$ *with* $\pi_i \in \Xi_i$ *such that* $\Xi \subseteq \alpha(\Xi)$.

Prop. 1(i) establishes existence, but not uniqueness, of each player's fully permissible set(s). Games with multiple strict Nash equilibria illustrate the possibility of such multiplicity; by Prop. 1(iii), any strict Nash equilibrium corresponds to a vector of fully permissible sets. Another (quite different) example of a game with multiple fully permissible sets is provided by G_5 of Sect. 5. Prop. 1(ii) means that Π is a fixed point in terms of a collection of vectors of strategy sets. By Prop. 1(iii) it is the largest such fixed point.

We close this section by recording some connections between IECFA on the one hand, and IESDS, the DF-procedure and IEWDS on the other. First, we note through the following Prop. 2 that IECFA has more bite than the DF procedure. G_1 of the introduction as well as G_2 of the present section illustrate that this refinement may be strict.

Proposition 2. *A pure strategy* p_i *is permissible (i.e., survives the DF procedure) if there exists a fully permissible set* π_i *such that* $p_i \in \pi_i$.

It follows as a corollary that IECFA has more cutting power also than IESDS, since a strategy is rationalizable (i.e. survives the IESDS) whenever it is permissible (i.e. survives the DF procedure).

	e	f	g
a	1, 1	1, 1	0, 0
b	1, 1	0, 1	1, 0
c	0, 1	0, 0	2, 0
d	0, 0	0, 1	0, 2

FIGURE 3. The relation between IECFA and IEWDS (G_3)

We finally compare IECFA to IEWDS. In both games discussed so far IECFA generates the same outcome as does IEWDS. However, the procedures of IECFA and IEWDS are different. This is even indicated in G_2, since — although IECFA and IEWDS have the same cutting power — the two algorithms work quite differently. In general, neither of IECFA and IEWDS has more bite than the other, as demonstrated by the game G_3 of Fig. 3. It is straightforward to verify that a and b for player 1, and e for player 2 survive IEWDS, while $\{a\}$ for 1 and $\{e, f\}$ for 2 survive IECFA and are thus the fully permissible sets, as shown below:

$$\begin{aligned}
\Xi(0) &= \Sigma_1 \times \Sigma_2 \\
\Xi(1) &= \{\{a\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\} \\
&\qquad\qquad\qquad \times \{\{e\}, \{g\}, \{e,f\}, \{e,g\}, \{f,g\}, \{e,f,g\}\} \\
\Xi(2) &= \{\{a\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\} \times \{\{e\}, \{e,f\}\} \\
\Xi(3) &= \{\{a\}, \{a,b\}\} \times \{\{e\}, \{e,f\}\} \\
\Xi(4) &= \{\{a\}, \{a,b\}\} \times \{\{e,f\}\} \\
\Pi = \Xi(5) &= \{\{a\}\} \times \{\{e,f\}\}
\end{aligned}$$

Strategy b survives IEWDS but does not appear in any fully permissible set. Strategy f appears in an fully permissible set but does not survive IEWDS. We refer to Sect. 5 for further comparison between IECFA and IEWDS.

3. States, Types, Preferences, and Belief

In the following two sections we provide an epistemic characterization of the concept of fully permissible sets. The first of these sections

- presents a framework for strategic games where each player is modeled as a decision maker under uncertainty, and
- introduces the epistemic operators that will be employed in this characterization.

The decision-theoretic analysis builds on Blume, Brandenburger & Dekel [15]. We relax *continuity* of preferences to allow the imposition of 'caution', as discussed in the introduction. Moreover, we also relax *completeness* of preferences to accommodate preferences that can be expressed solely in terms of admissibility on nested sets, and thus cannot be represented by means of subjective probabilities. The framework is summarized by the concept of a *belief system* (cf. Def. 2). Appendix A contains a presentation of the decision-theoretic terminology, notation and results that will be utilized.

3.1. **A Strategic Game Form.** Let $z : S \to Z$ map strategy vectors into *outcomes*, where Z is the set of outcomes. Then $((S_i)_{i \in N}, z)$ is a finite *strategic* two-player *game form*.

3.2. **States and Types.** When a strategic game form is turned into a decision problem for each player (see Tan & Werlang [39]), the uncertainty faced by a player concerns the strategy choice of his opponent, the belief of his opponent about his own strategy choice, and so on. A type of a player corresponds to a vNM utility function and a belief about the strategy choice of his opponent, a belief about the belief of his opponent about his own strategy choice, and so on.

Given an assumption of coherency, models of such infinite hierarchies of beliefs (Armbruster & Böge [2], Böge & Eisele [17], Mertens & Zamir [33], Brandenburger & Dekel [22], Epstein & Wang [28]) yield $S \times T$ as the complete state space, where S is the underlying space of uncertainty and where $T = T_1 \times T_2$ is the set of all feasible type vectors. Furthermore, for each i, there is a homeomorphism between T_i and the set of beliefs on $S \times T_j$, where j denotes i's opponent. Combined with a vNM utility function, the set of beliefs on $S \times T_j$ corresponds to the set of "regular" binary relations on the set of acts on $S \times T_j$, where an *act* on $S \times T_j$ is a function that to any element of $S \times T_j$ assigns an objective randomization on Z.

For each type of any player i, the type's decision problem is to choose one of i's strategies. For the modeling of this problem, the type's belief about his own decision is not relevant and can be ignored. Hence, models of infinite hierarchies of beliefs — in the setting of a strategic game form — imply that each type of any player i corresponds to a "regular" binary relation on the set of acts on $S_j \times T_j$.

In conformity with the literature on infinite hierarchies of beliefs, let

- the set of *states of the world* (or simply *states*) be $\Omega := S \times T$,
- each *type* t_i of any player i correspond to a binary relation $\succeq^{t_i}$ on the set of acts on $S_j \times T_j$.

However, we do not construct a complete state space by explicitly modeling infinite hierarchies of beliefs. For tractability we instead directly consider an implicit model — with a finite type set T_i for each player i — from which infinite hierarchies of beliefs can be constructed.[1] Moreover, since completeness and continuity of preferences are not imposed, the "regularity" conditions on $\succeq^{t_i}$ consist of *reflexivity*, *transitivity*, *objective independence*, *nontriviality*, *conditional completeness*, *conditional continuity* and *non-null*

[1] This is not purely a matter of convenience as Brandenburger [21] and Brandenburger & Keisler [23] have shown that a complete state space may not exist if beliefs are not based on subjective probabilities. In contrast to Battigalli & Siniscalchi's [11] epistemic foundation for EFR, a complete state space is not needed for the present analysis.

state independence, meaning that $\succeq^{t_i}$ is conditionally represented by a vNM utility function $\upsilon_i^{t_i} : Z \to \mathbb{R}$ that assigns a payoff to any outcome (cf. Prop. A1 of Appendix A).[2] Being a vNM utility function, $\upsilon_i^{t_i}$ can be extended to objective randomizations on Z. Since $\succeq^{t_i}$ is conditionally represented, it follows that strong and weak dominance are well-defined. The construction is summarized by the following definition.

Definition 2. A *belief system* for a game form $((S_i)_{i\in N}, z)$ consists of

- for each player i, a finite set of types T_i,
- for each type t_i of any player i, a binary relation $\succeq^{t_i}$ (t_i's *preferences*) on the set of acts on $S_j \times T_j$, where $\succeq^{t_i}$ is conditionally represented by a vNM utility function $\upsilon_i^{t_i}$.

3.3. **Belief and Certain Belief.** When preferences are not continuous, one can differentiate between belief and certain belief in a manner that will be explained below. Both 'belief' and 'certain belief' are subjective, as they are derived from preferences (following the approach of Morris [34]); hence, neither operator satisfies the truth axiom. To state these operators, let, for each player i and each state $\omega \in \Omega$, $t_i(\omega)$ denote the projection of ω on T_i, and let, for any event $E \subseteq \Omega$, $E_j^{t_i} := \{(s_j, t_j) \in S_j \times T_j | \exists (s_1', s_2', t_1', t_2') \in E$ s.t. $(s_j', t_j') = (s_j, t_j)$ and $t_i' = t_i\}$ denote the set of opponent strategy-type pairs that are consistent with $\omega \in E$ and $t_i(\omega) = t_i$.

It is perhaps easier to introduce these concepts in the case when preferences are complete and, thus, representable in terms of an LPS $\lambda^{t_i} = (\mu_1^{t_i}, \ldots, \mu_L^{t_i}) \in \mathbf{L}\Delta(S_j \times T_j)$ (cf. footnote 2). Then an event is 'certainly believed' if no element of the complement is assigned positive probability by some probability distribution in λ^{t_i}:

$$K_i E := \{\omega \in \Omega | \kappa_j^{t_i(\omega)} \subseteq E_j^{t_i(\omega)}\},$$

where $\kappa_j^{t_i} := \mathrm{supp}\lambda^{t_i}$ ($\subseteq S_j \times T_j$). On the other hand, an event is 'believed' if no element of the complement is assigned positive probability by $\mu_1^{t_i}$:[3]

$$B_i E := \{\omega \in \Omega | \beta_j^{t_i(\omega)} \subseteq E_j^{t_i(\omega)}\},$$

where $\beta_j^{t_i} := \mathrm{supp}\mu_1^{t_i}$ ($\subseteq S_j \times T_j$). It follows that $K_i E \subseteq B_i E$ (i.e. 'certain belief' implies 'belief') since $\beta_j^{t_i} = \mathrm{supp}\mu_1^{t_i} \subseteq \kappa_j^{t_i} = \mathrm{supp}\lambda^{t_i} := \cup_{\ell=1}^{L} \mathrm{supp}\mu_\ell^{t_i}$.

---

[2]If conditional completeness is strengthened to completeness, then it follows from Blume, Brandenburger & Dekel [15] that $\succeq^{t_i}$ is represented by $\upsilon_i^{t_i}$ and a lexicographic probability system (LPS) $\lambda^{t_i} = (\mu_1^{t_i}, \ldots, \mu_L^{t_i}) \in \mathbf{L}\Delta(S_j \times T_j)$ (cf. Prop. A2 of Appendix A). If, in addition, conditional continuity is strengthened to continuity, then $\succeq^{t_i}$ is represented by $\upsilon_i^{t_i}$ and a subjective probability distribution $\mu^{t_i} \in \Delta(S_j \times T_j)$. Continuity is inconsistent with the present analysis due to the requirement of 'caution'. Completeness, implying a subjective probability representation through an LPS, is consistent with – though not implied by – the concept of 'admissible consistency', but inconsistent with the concept of 'full admissible consistency' (cf. Sect. 4).

[3]This notion of 'belief' in the case of complete preferences corresponds to Brandenburger's [20] 'first-order knowledge'.

To generalize $\kappa_j^{t_i}$ (and thus K_iE) to incomplete preferences, let

$$\kappa_j^{t_i} := \{(s_j, t_j) \in S_j \times T_j | (s_j, t_j) \text{ is not Savage-null acc. to } \succeq^{t_i}\}$$

denote the set of opponent strategy-type pairs that t_i deems possible.[4] This generalizes the case of complete preferences, since in that case $\text{supp}\lambda^{t_i}$ is the set of opponent strategy-type pairs that t_i does not deem Savage-null.

To generalize $\beta_j^{t_i}$ (and thus B_iE) to incomplete preferences, say that $\succeq^{t_i}$ *is admissible on* β_j, where $\emptyset \neq \beta_j \subseteq S_j \times T_j$, if $\mathbf{x} \succ^{t_i} \mathbf{y}$ whenever $\mathbf{x}_{\beta_j}$ weakly dominates $\mathbf{y}_{\beta_j}$. If $\succeq^{t_i}$ is admissible on β_j, then any $(s_j', t_j') \in \beta_j$ is deemed infinitely more likely than any $(s_j'', t_j'') \in S_j \times T_j \backslash \beta_j$. Since (s_j', t_j') being infinitely more likely than (s_j'', t_j'') implies that (s_j'', t_j'') is *not* infinitely more than (s_j', t_j'), it follows that $\beta_j' \subseteq \beta_j''$ or $\beta_j' \supseteq \beta_j''$ whenever $\succeq^{t_i}$ is admissible on both β_j' and β_j''. Since, in addition, $\succeq^{t_i}$ is admissible on $\kappa_j^{t_i}$, it follows that there exists a unique smallest (w.r.t. set inclusion) non-empty set on which $\succeq^{t_i}$ is admissible; let this set be denoted $\beta_j^{t_i}$:[5]

$$\succeq^{t_i} \text{ is admissible on } \beta_j^{t_i} \text{ and } \beta_j \supseteq \beta_j^{t_i} \text{ whenever } \succeq^{t_i} \text{ is admissible on } \beta_j\,.$$

This generalizes the case of complete preferences, since in that case $\text{supp}\mu_1^{t_i}$ is the unique smallest set of opponent strategy-type pairs on which $\succeq^{t_i}$ is admissible. Also with incomplete preferences it follows that $K_iE \subseteq B_iE$ since $\succeq^{t_i}$ is admissible on $\kappa_j^{t_i}$; i.e. $\beta_j^{t_i} \subseteq \kappa_j^{t_i}$. If $\beta_j^{t_i} \neq \kappa_j^{t_i}$, then t_i's preferences are not continuous.

In addition to $K_iE \subseteq B_iE$, it follows that the operators B_i and K_i satisfy

$$\begin{aligned} B_iE \cap B_iF &= B_i(E \cap F) & \qquad K_iE \cap K_iF &= K_i(E \cap F) \\ B_i\emptyset &= \emptyset & K_i\Omega &= \Omega \\ B_iE &\subseteq K_iB_iE & K_iE &\subseteq K_iK_iE \\ \neg B_iE &\subseteq K_i(\neg B_iE) & \neg K_iE &\subseteq K_i(\neg K_iE). \end{aligned}$$

Since $K_iE \subseteq B_iE$ implies that $K_i\emptyset = \emptyset$, $B_i\Omega = \Omega$, $B_iE \subseteq B_iB_iE$ and $\neg B_iE \subseteq B_i(\neg B_iE)$, both operators B_i and K_i correspond to KD45 systems. Since an event can be certainly believed even though the true state is an element of the complement of the event, it follows that neither operator satisfies the truth axiom (i.e. $K_iE \subseteq E$ and $B_iE \subseteq E$ need not hold).

Say that i believes the event $E \subseteq \Omega$ given ω if $\omega \in B_iE$ (or equivalently, $\beta_j^{t_i(\omega)} \subseteq E_j^{t_i(\omega)}$). Say that i certainly believes the event $E \subseteq \Omega$ given ω if $\omega \in K_iE$ (or equivalently, $\kappa_j^{t_i(\omega)} \subseteq E_j^{t_i(\omega)}$). Write $KE := K_1E \cap K_2E$. Say that there is mutual certain belief of $E \subseteq \Omega$ given ω if $\omega \in KE$. Write $CKE := KE \cap KKE \cap KKKE \cap \dots$. Say that there is common certain belief of $E \subseteq \Omega$ given ω if $\omega \in CKE$.

[4]The term 'certain belief' for this notion is also used by Morris [34].

[5]This notion of 'belief' is related to, but differs from, Morris' [34] 'strong belief'.

3.4. **Full Belief.** In Sect. 4 the concept of fully permissible sets will be characterized by imposing common certain belief of the event of 'full admissible consistency'. The definition of 'full admissible consistency' is based on an epistemic operator that we will refer to as 'full belief'. This subsection introduces and characterizes this operator.

An event is 'fully believed' if any element of the event that is not Savage-null is deemed infinitely more likely than any element of the complement:

$$B_i^0 E := \{\omega \in \Omega | \succeq^{t_i(\omega)} \text{ is admissible on } E_j^{t_i(\omega)} \cap \kappa_j^{t_i(\omega)}\}.$$

It follows that $B_i^0 E \subseteq B_i E$ since $\succeq^{t_i(\omega)}$ being admissible on $E_j^{t_i(\omega)} \cap \kappa_j^{t_i(\omega)}$ implies that $\beta_j^{t_i(\omega)} \subseteq E_j^{t_i(\omega)}$. It follows that $K_i E \subseteq B_i^0 E$ since $\kappa_j^{t_i(\omega)} \subseteq E_j^{t_i(\omega)}$ implies that $\succeq^{t_i(\omega)}$ is admissible on $\kappa_j^{t_i(\omega)} = E_j^{t_i(\omega)} \cap \kappa_j^{t_i(\omega)}$.

In addition to $K_i E \subseteq B_i^0 E \subseteq B_i E$, it follows that the operator B_i^0 satisfies

$$B_i^0 E \cap B_i^0 F \subseteq B_i^0 (E \cap F)$$
$$B_i^0 E \subseteq K_i B_i^0 E$$
$$\neg B_i^0 E \subseteq K_i (\neg B_i^0 E).$$

Note that $K_i E \subseteq B_i^0 E \subseteq B_i E$ implies that $B_i^0 \emptyset = \emptyset$, $B_i^0 \Omega = \Omega$, $B_i^0 E \subseteq B_i^0 B_i^0 E$ and $\neg B_i^0 E \subseteq B_i^0 (\neg B_i^0 E)$. However, even though the operator B_i^0 satisfies $B_i^0 E \subseteq \neg B_i^0 \neg E$ as well as positive and negative introspection, it does not satisfy monotonicity since $E \subseteq F$ does not imply $B_i^0 E \subseteq B_i^0 F$. To see that the operator B_i^0 does not satisfy monotonicity, consider G_1 of the introduction: If 2 prefers any strategy that (weakly) dominates another on $\{U\}$, regardless of what happen outside $\{U\}$, then it does not follow that 2 prefer any strategy that weakly dominates another on $\{U, M\}$, regardless of what happens outside $\{U, M\}$, since weak dominance on $\{U, M\}$ does not imply (weak) dominance on $\{U\}$. This is illustrated by L and R: $\omega \in B_2^0(\{(s_1, s_2, t_1, t_2) | s_1 = U\})$ does *not* imply that $t_2(\omega)$ prefers L to R, while $\omega \in B_2^0(\{(s_1, s_2, t_1, t_2) | s_1 \in \{U, M\}\})$ *does* imply that $t_2(\omega)$ prefers L to R.

Such non-monotonic operators arise also in other contributions that provide epistemic conditions for forward induction. In particular, Battigalli & Siniscalchi [11] use a non-monotonic operator which they call 'strong belief'. However, in contrast to Battigalli & Siniscalchi's [11] use of the operator 'strong belief', our non-monotonic operator B_i^0 is used *only* for defining the event of 'full admissible consistency', while the monotonic operator K_i is used for the *interactive* epistemology.

Say that i fully believes the event $E \subseteq \Omega$ given ω if $\omega \in B_i^0 E$ (or equivalently, $\succeq^{t_i(\omega)}$ is admissible on $E_j^{t_i(\omega)} \cap \kappa_j^{t_i(\omega)}$).

3.5. **Preferences over Strategies.** Let $\succeq_{S_j}^{t_i}$ denote the *marginal* of $\succeq^{t_i}$ on S_j. A pure strategy $s_i \in S_i$ can be viewed as an act $\mathbf{x}_{S_j}$ on S_j that assigns $z(s_i, s_j)$ to any $s_j \in S_j$. A *mixed* strategy $x_i \in \Delta(S_i)$ corresponds to an act $\mathbf{x}_{S_j}$ on S_j that assigns $z(x_i, s_j)$ to any $s_j \in S_j$. Hence, $\succeq_{S_j}^{t_i}$ is a binary relation also on the subset of acts on S_j that correspond to i's mixed strategies. Thus, $\succeq_{S_j}^{t_i}$ can be referred to as t_i's *preferences over i's mixed* *strategies*. The set of mixed strategies $\Delta(S_i)$ is the set of acts that are at t_i's actual disposal.

Since $\succeq^{t_i}$ is reflexive and transitive and satisfies objective independence, $\succeq^{t_i}_{S_j}$ shares these properties, and t_i's *choice set*,

$$C_i^{t_i} := \{s_i \in S_i | s_i \text{ is maximal w.r.t. } \succeq^{t_i}_{S_j} \text{ in } \Delta(S_i)\},$$

is non-empty and supports any maximal mixed strategy.

3.6. **Playing the Game.** The event that i plays the game $G = (S_i, u_i)_{i\in N}$ is given by

$$[u_i] := \left\{\omega \in \Omega | v_i^{t_i(\omega)} \circ z \text{ is a positive affine transformation of } u_i\right\},$$

while $[u_1] \cap [u_2]$ is the event that both players play G.

4. Consistency of Preferences

Usually requirements in deductive game theory are imposed on choice. E.g. rationality is a requirement on a pair (s_i, t_i), where s_i is said to be a 'rational choice' by t_i if $s_i \in C_i^{t_i}$, and where the event that i is rational is defined as[6]

$$[rat_i] := \left\{(s_1, s_2, t_1, t_2) \in \Omega | s_i \in C_i^{t_i}\right\}.$$

The present paper imposes requirements on t_i only. Since t_i corresponds to the preferences $\succeq^{t_i}$, such requirements will be imposed on $\succeq^{t_i}$. In support of this alternative approach — which will be referred to by the term 'consistent preferences' — one can note the following: The approach allows

- ... requirements to be imposed on types rather than strategy-type pairs.
- ... conventional concepts like rationalizable and permissible strategies to be characterized under weak and natural conditions (see e.g. Prop. 3 and Remark 1 below).
- ... requirements like 'caution' and 'full belief of opponent rationality' to be imposed in a straightforward manner. Under the usual approach, the notion of 'certain belief' must be weakened to accommodate caution (cf. Börgers ([18], pp. 266–267) and Epstein ([27], p. 3)), and a non-monotonic epistemic operator must be used for the interactive epistemology to accommodate full belief of opponent rationality.

Here we will focus on showing how 'consistent preferences' as an approach to deductive game-theoretic analysis can be used to provide an epistemic characterization for the algorithm presented in Sect. 2 and, thus, to provide epistemic conditions for aspects of forward induction. In particular, we will

1. ... characterize the concept of permissible strategies through imposing common certain belief of 'admissible consistency'. This exercise has separate interest since it differs from the results of Börgers [18] and Brandenburger [20] by providing an epistemic foundation for the DF

[6] See e.g. Epstein ([27], Sect. 6) for a presentation of this approach in a general context.

procedure by means of an operator (certain belief) that need *not* be weakened to allow the complement of a believed event to be taken into account.

2. ... argue for the alternative and stronger requirement of 'full admissible consistency', and characterize the main concept of this paper – fully permissible sets – by imposing common certain belief of 'full admissible consistency'.

4.1. **Admissible Consistency.** Below we characterize the concept of permissible strategies in a finite strategic game $G = (S_i, u_i)_{i \in N}$ by imposing three requirements: The first of these ensures that each player plays the game G, the second requirement ensures that each player takes all opponent strategies into account ('caution'), while the third requirement ensures that each player believes that the opponent chooses rationally ('belief of opponent rationality'). The first requirement is stated in subsect. 3.6. To impose the other two, consider the following events

$$[cau_i] := \{\omega \in \Omega | \kappa_j^{t_i(\omega)} = S_j \times T_j^{t_i(\omega)}\}$$
$$B_i[rat_j] = \{\omega \in \Omega | \beta_j^{t_i(\omega)} \subseteq [rat_j]_j\} ,$$

where $T_j^{t_i} := \mathrm{proj}_{T_j} \kappa_j^{t_i}$ denotes the set of opponent types that t_i deems possible, and where $[rat_j]_j := \mathrm{proj}_{S_j \times T_j}[rat_j] = \{(s_j, t_j) | s_j \in C_j^{t_j}\}$.[7]

- If $\omega \in [cau_i]$, then (s_j, t_j) is deemed possible according to $\succeq^{t_i(\omega)}$ whenever t_j is deemed possible. This means that, $\forall (s_j, t_j) \in S_j \times T_j^{t_i(\omega)}$, $\omega \notin K_i\{(s_1', s_2', t_1', t_2') \in \Omega | (s_j', t_j') \neq (s_j, t_j)\}$ (cf. Dekel & Gul's [25] definition of caution). It implies that the marginal of $\succeq^{t_i(\omega)}$ on S_j (i.e., $t_i(\omega)$'s preferences over S_i, $\succeq_{S_j}^{t_i(\omega)}$) is admissible on S_j.
- If $\omega \in B_i[rat_j]$, then i believes given ω that j is rational.

Say that i is *admissibly consistent* (with the game G and the preferences of his opponent) given ω if $\omega \in A_i$, where

$$A_i := [u_i] \cap [cau_i] \cap B_i[rat_j].$$

Refer to $A := A_1 \cap A_2$ as the event of *admissible consistency.* We can now characterize the concept of permissible strategies as maximal strategies in states where there is common certain belief of admissible consistency.

Proposition 3. *A pure strategy p_i for i is permissible in a finite strategic game G if and only if there exists a belief system with $p_i \in C_i^{t_i(\omega)}$ for some $\omega \in CKA$.*

Remark 1. The concept of rationalizable strategies in 2-player games can be characterized by removing the requirement of 'caution': A pure strategy r_i

[7] If $\omega \in [cau_i] \cap B_i[rat_j]$ and $\succeq^{t_i(\omega)}$ is complete, then $\succeq^{t_i(\omega)}$ can be represented by $v_i^{t_i(\omega)}$ and an LPS $\lambda^{t_i(\omega)} = (\mu_1^{t_i(\omega)}, \dots, \mu_L^{t_i(\omega)}) \in \mathbf{L}\Delta(S_j \times T_j)$ satisfying $\mathrm{supp}\lambda^{t_i(\omega)} = S_j \times T_j^{t_i(\omega)}$ and $\mu_1^{t_i(\omega)}(r_j, t_j) > 0$ only if $r_j \in C_j^{t_j}$. Note that $\omega \in [cau_i] \cap B_i[rat_j]$ does not imply — but is consistent with — $\succeq^{t_i(\omega)}$ being complete, while $\omega \in [cau_i] \cap B_i[rat_j]$ is *not* consistent with $\succeq^{t_i(\omega)}$ being continuous.

for i is rationalizable in a finite strategic game G if and only if there exists a belief system with $r_i \in C_i^{t_i(\omega)}$ for some $\omega \in CK([u_1] \cap B_1[rat_2] \cap [u_2] \cap B_2[rat_1])$.

Remark 2. It turns out that Prop. 3 holds even if the additional requirement of minimal completeness is imposed. In fact, the proof in Appendix B applies for the following result: A pure strategy p_i for i is permissible in a finite strategic game G if and only if there exists a belief system with $p_i \in C_i^{t_i(\omega)}$ for some $\omega \in CK\bar{A}$, where $\bar{A} := \bar{A}_1 \cap \bar{A}_2$, and where, for each i,

$$\begin{aligned}\bar{A}_i := [u_i] \ \cap \ \{\omega \in \Omega | \mathbf{x} \succ^{t_i(\omega)} \mathbf{y} \text{ if and only if } \mathbf{x}_{\beta_j} \text{ weakly dom. } \mathbf{y}_{\beta_j} \\ \text{for } \beta_j = \beta_j^{t_i(\omega)} \subseteq [rat_j]_j \text{ or } \beta_j = \kappa_j^{t_i(\omega)} = S_j \times T_j^{t_i(\omega)}\}.\end{aligned}$$

Hence, imposing minimal completeness does not alone yield a refinement.

4.2. **Full Admissible Consistency.** We propose that it is natural to substitute the requirement 'full belief of opponent rationality' for 'belief of opponent rationality'. Therefore, consider the following events.

$$\begin{aligned}[cau_i] &:= \{\omega \in \Omega | \kappa_j^{t_i(\omega)} = S_j \times T_j^{t_i(\omega)}\} \\ B_i^0[rat_j] &= \{\omega \in \Omega | \succeq^{t_i(\omega)} \text{ is admissible on } [rat_j]_j \cap \kappa_j^{t_i(\omega)}\}\,.\end{aligned}$$

If $\omega \in B_i^0[rat_j]$, then i fully believes given ω that j is rational. This means that any (s_j', t_j') which is deemed possible and where s_j' *is* a rational choice by t_j' is considered infinitely more likely than any (s_j'', t_j'') where s_j'' is *not* a rational choice by t_j''. Write $A^0 := A_1^0 \cap A_2^0$, where for each i,

$$A_i^0 := [u_i] \cap [cau_i] \cap B_i^0[rat_j]\,.$$

Note that $A^0 \subseteq A$ since, for each i, $B_i^0[rat_j] \subseteq B_i[rat_j]$.

To motivate the strengthening of 'belief of opponent rationality' to 'full belief of opponent rationality' (and, thus, of A to A^0), return to G_1 of the introduction. In this game, $\{U, M\}$ is the choice set of any type of player 1 (provided that $\omega \in [u_1] \cap [cau_1]$). While 'belief of opponent rationality' is consistent with (some type of) player 2 *not* deeming M more likely than the remaining non-maximal strategy D and, thus, *not* preferring L to R, 'full belief of opponent rationality' ensures that M is deemed infinitely more likely than D. Hence, if $\omega \in CKA^0$, only L is a maximal strategy for $t_2(\omega)$.

However, common certain belief of the event A^0 is not sufficient to promote the forward induction outcome in G_2 of Sect. 2. To see this, consider a belief system with only one type of each player; i.e., $T_1 \times T_2 = \{t_1\} \times \{t_2\}$. Let, for each i, $\succeq_i^{t_i}$ satisfy that $v_i^{t_i} \circ z = u_i$. Let t_1 deem (R, t_2) infinitely more likely than (L, t_2), with (L, t_2) not being Savage-null. Then $C_1^{t_1} = \{U\}$. Let t_2 deem (U, t_1) infinitely more likely than (D, t_1) and (D, t_1) infinitely more likely than (M, t_1), with (M, t_1) not being Savage-null. Then $C_2^{t_2} = \{R\}$. Inspection will verify that $CKA^0 = A^0 = \Omega = S \times T_1 \times T_2$. Hence, strengthening A to A^0 is not sufficient to promote the forward induction outcome (M, L) in this game. In fact, we have the following general result.

Proposition 4. *If* $(x_1, x_2) \in \Delta(S_1) \times \Delta(S_2)$ *is a proper equilibrium in a finite strategic game* G*, then, for each* i *and any* $s_i \in \mathrm{supp} x_i$*, there exists a belief system with* $s_i \in C_i^{t_i(\omega)}$ *for some* $\omega \in CKA^0$*.*

Note that (U, R) is a proper equilibrium in G_2. However, the preferences of t_2 in the belief system for G_2 above are not minimally complete in the sense of being characterized by 'caution' and 'full belief of opponent rationality'. In particular, t_2 deems one of t_1's non-maximal strategies, D, infinitely more likely than the another non-maximal strategy, M. It turns out that the additional imposition of minimal completeness leads to a characterization of the concept of fully permissible sets, and, hence, to the promotion of the forward induction outcome in G_2 (cf. the analysis of G_2 in Sects. 2 and 5).

To impose minimal completeness, consider for each i,

$$\begin{aligned}\bar{A}_i^0 := [u_i] \cap \{\omega \in \Omega | \mathbf{x} \succ^{t_i(\omega)} \mathbf{y} &\text{ if and only if } \mathbf{x}_{\beta_j} \text{ weakly dom. } \mathbf{y}_{\beta_j} \\ &\text{for } \beta_j = \beta_j^{t_i(\omega)} = [rat_j]_j \cap \kappa_j^{t_i(\omega)} \text{ or } \beta_j = \kappa_j^{t_i(\omega)} = S_j \times T_j^{t_i(\omega)}\},\end{aligned}$$

where it follows that $\bar{A}_i^0 \subseteq [u_i] \cap [cau_i] \cap B_i^0[rat_j] = A_i^0$. Say that i is *fully admissibly consistent* (with the game G and the preferences of his opponent) given ω if $\omega \in \bar{A}_i^0$, and refer to $\bar{A}^0 := \bar{A}_1^0 \cap \bar{A}_2^0$ as the event of *full admissible consistency.* We can now characterize the main concept of the present paper – fully permissible sets – as choice sets in states where there is common certain belief of full admissible consistency.

Proposition 5. *A non-empty strategy set* π_i *for* i *is fully permissible in a finite strategic game* G *if and only if there exists a belief system with* $\pi_i = C_i^{t_i(\omega)}$ *for some* $\omega \in CK\bar{A}^0$*.*

Remark 3. Since Prop. 3 holds even if the additional requirement of minimal completeness is imposed (cf. Remark 2), it follows from Prop. 5 that the refinement relative to permissible strategies – offered by the concept of fully permissible sets – is effectively due to strengthening of the requirement of $\beta_j^{t_i(\omega)} \subseteq [rat_j]_j \cap \kappa_j^{t_i(\omega)}$ to the requirement of $\beta_j^{t_i(\omega)} = [rat_j]_j \cap \kappa_j^{t_i(\omega)}$.

5. Further examples

The present section illustrates the concept of fully permissible sets by returning to the previously discussed game G_2 as well as by considering two new examples. Of the three examples, the two first will be used to show how our concept captures aspects of forward induction, while the last example will illustrate the possibility of multiple fully permissible sets.

All three examples will be used to shed light on the differences between the approach suggested here and IEWDS (where at each round all weakly dominated strategies are eliminated). This comparative discussion will be facilitated if we can refer to a characterization of IEWDS established by Stahl [38]: Assume that each player, for each round g, deems *any* opponent strategy not yet eliminated by round g infinitely more likely that any eliminated strategy. Hence, the player's belief will have a hierarchical structure

	LL	LR	RL	RR
NU	3, 1	3, 1	0, 0	0, 0
ND	0, 0	0, 0	1, 3	1, 3
BU	2, 1	-1, 0	2, 1	-1, 0
BD	-1, 0	0, 3	-1, 0	0, 3

FIGURE 4. Burning money (G_4)

if opponent strategies are eliminated through several rounds. Then a strategy survives IEWDS if and only if it is maximal w.r.t. preferences that are compatible with these hierarchical beliefs.[8]

5.1. **Forward Induction.** Reconsider G_2 of Sect. 2, and apply our algorithm IECFA to this 'Battle-of-the-sexes-with-an-outside-option' game. Since D is a strongly dominated strategy, D cannot be an element of 1's choice set. This *does not* imply, as in the procedure of IEWDS (given Stahl's [38] characterization), that 2 deems M infinitely more likely than D. However, 2 certainly believes that only $\{U\}$, $\{M\}$ and $\{U, M\}$ are candidates for 1's choice set. This excludes $\{R\}$ as 2's choice set, since $\{R\}$ is 2's choice set only if 2 deems $\{D\}$ or $\{U, D\}$ possible. This in turn means that 1 certainly believes that only $\{L\}$ and $\{L, R\}$ are candidates for 2's choice set, implying that $\{U\}$ cannot be 1's choice set. Certainly believing that only $\{M\}$ and $\{U, M\}$ are candidates for 1's choice set *does* imply that 2 deems M infinitely more likely than D. Hence, 2's choice set is $\{L\}$ and, therefore, 1's choice set $\{M\}$. Thus, the forward induction outcome (M, L) is promoted.

Turn now to the 'Burning money' game due to van Damme ([40], Fig. 5) and Ben-Porath & Dekel ([13], Fig. 1.2). G_4 of Fig. 4 is the pure strategy reduced strategic form of a 'Battle-of-the-sexes' (B-o-s) game with the additional feature that 1 can publicly destroy 1 unit of payoff before the B-o-s game starts. BU (NU) is the strategy where 1 burns (does not burn), and then plays U, etc., while LR is the strategy where 2 responds with L conditional on 1 not burning and R conditional on 1 burning, etc. The forward induction outcome (supported e.g. by IEWDS) involves implementation of 1's preferred B-o-s outcome, with *no payoff being burnt*.

One might be skeptical to the use of IEWDS in the 'Burning money' game, because it effectively requires 2 to infer that BU is infinitely more likely than BD based on the sole premise that BD is eliminated before BU, even though all strategies involving burning (i.e. both BU and BD) are eventually eliminated by the procedure. On the basis of this premise such an inference seems at best to be questionable. As shown in Table 1, the application of our algorithm IECFA yields an iteration where at no stage

[8]Battigalli [9] establishes a related result. See also Rajan [36]. Brandenburger's [20] analysis implies that the DF procedure can be characterized by assuming that each player, for each round g, deems *some* opponent strategy not yet eliminated by round g infinitely more likely that any eliminated strategy.

$$
\begin{aligned}
\Xi(0) &= \Sigma_1 \times \Sigma_2 \\
\Xi(1) &= \{\{NU\},\{ND\},\{BU\},\{NU,ND\},\{ND,BU\},\{NU,BU\},\{NU,ND,BU\}\} \times \Sigma_2 \\
\Xi(2) &= \{\{NU\},\{ND\},\{BU\},\{NU,ND\},\{ND,BU\},\{NU,BU\},\{NU,ND,BU\}\} \\
&\qquad \times \{\{LL\},\{RL\},\{LL,LR\},\{RL,RR\},\{LL,RL\},\{LL,LR,RL,RR\}\} \\
\Xi(3) &= \{\{NU\},\{BU\},\{ND,BU\},\{NU,BU\},\{NU,ND,BU\}\} \\
&\qquad \times \{\{LL\},\{RL\},\{LL,LR\},\{RL,RR\},\{LL,RL\},\{LL,LR,RL,RR\}\} \\
\Xi(4) &= \{\{NU\},\{BU\},\{ND,BU\},\{NU,BU\},\{NU,ND,BU\}\} \\
&\qquad \times \{\{LL\},\{RL\},\{LL,LR\},\{LL,RL\}\} \\
\Xi(5) &= \{\{NU\},\{BU\},\{NU,BU\}\} \times \{\{LL\},\{RL\},\{LL,RL\},\{LL,RL\}\} \\
\Xi(6) &= \{\{NU\},\{BU\},\{NU,BU\}\} \times \{\{LL\},\{LL,LR\},\{LL,RL\}\} \\
\Xi(7) &= \{\{NU\},\{NU,BU\}\} \times \{\{LL\},\{LL,LR\},\{LL,RL\}\} \\
\Xi(8) &= \{\{NU\},\{NU,BU\}\} \times \{\{LL\},\{LL,LR\}\} \\
\Xi(9) &= \{\{NU\}\} \times \{\{LL\},\{LL,LR\}\} \\
\Pi = \Xi(10) &= \{\{NU\}\} \times \{\{LL,LR\}\}
\end{aligned}
$$

TABLE 1. Applying IECFA to 'Burning money'.

	L	R
U	1, 1	1, 1
M	0, 1	2, 0
D	1, 0	0, 1

FIGURE 5. Game with multiple fully permissible sets (G_5).

need 2 deem BU infinitely more likely than BD since $\{NU\}$ is always included as a candidate for 1's choice set. The procedure uniquely determines $\{NU\}$ as 1's fully permissible set and $\{LL, LR\}$ as 2's fully permissible set.[9] Even though the forward induction *outcome* is obtained, 2 does not have any assessment concerning the relative likelihood of opponent strategies conditional on burning; hence, she need not interpret burning as a signal that 1 will play according with his preferred B-o-s outcome.

We can conclude that the concept of fully permissible sets yields the forward induction outcome in G_2 and G_4. Furthermore, the concept promotes forward induction for different reasons than does the procedure of IEWDS (and the concept of EFR, which works like IEWDS in these games).

5.2. **Multiple Fully Permissible Sets.** In G_5 of Fig. 5, IEWDS eliminates D in the first round, R in the second round, and M in the third round, so that U and L survive. Stahl's [38] characterization of IEWDS entails that 2 deems *each* of U and M infinitely more likely than D. Hence, the procedure forces 2 to deem M infinitely more likely than D for the sole reason

[9] Also Battigalli [8], Asheim [3], and Dufwenberg [26] (as well as Hurkens [31] in a slightly different context) argue that (NU, LR) in addition to (NU, LL) is a viable strategy vector in 'Burning money'.

that D is eliminated before M, even though both M and D are eventually eliminated by the procedure.

Turn now to IECFA, which yields:

$$\begin{aligned}
\Xi(0) &= \Sigma_1 \times \Sigma_2 \\
\Xi(1) &= \{\{U\}, \{M\}, \{U, M\}\} \times \Sigma_2 \\
\Xi(2) &= \{\{U\}, \{M\}, \{U, M\}\} \times \{\{L\}, \{L, R\}\} \\
\Pi = \Xi(3) &= \{\{U\}, \{U, M\}\} \times \{\{L\}, \{L, R\}\}
\end{aligned}$$

Since D is a weakly dominated strategy, D cannot be an element of 1's choice set. Hence, 2 certainly believes that only $\{U\}$, $\{M\}$ and $\{U, M\}$ are candidates for 1's choice set. This excludes $\{R\}$ as 2's choice set, since $\{R\}$ is 2's choice set only if 2 deems $\{D\}$ or $\{U, D\}$ possible. This in turn means that 1 certainly believes that only $\{L\}$ and $\{L, R\}$ are candidates for 2's choice set, implying that $\{M\}$ cannot be 1's choice set. There is no further elimination. This means that 1's collection of fully permissible sets is $\{\{U\}, \{U, M\}\}$ and 2's collection of fully permissible sets is $\{\{L\}, \{L, R\}\}$. Thus, common certain belief of full admissible consistency implies that 2 deems U infinitely more likely than D since U (respectively, D) is an element of any (respectively, no) fully permissible set for 1. However, whether 2 deems M infinitely more likely than D depends on the type of player 2.

Note that in G_5 there cannot be mutual certain belief of the players' choice sets. E.g. if 1's choice set is $\{U\}$, it is because 1 certainly believes that 2's choice set is $\{L\}$. However, $\{L\}$ is 2's choice set only if 2 does *not* certainly believe that 1's choice set is $\{U\}$. Likewise, if 1's choice set is $\{U, M\}$.

Multiplicity of fully permissible sets arises also in the strategic form of some well-known extensive games in which the application of backward induction has been subject to debate, e.g. the 'Centipede' game. See Asheim & Dufwenberg [6] for more on this.

6. Concluding remarks

We end by discussing related literature as well as commenting on the scope of the general approach chosen in the present paper.

6.1. **Related Literature.** It is instructive to explain how our analysis differs from the epistemic foundation of EFR provided by Battigalli & Siniscalchi [11]. An analogous comparison can be made to contributions that provide epistemic conditions for IEWDS, see e.g. Rajan [36] and Stahl [38]. It turns out to be of minor importance for the comparison to EFR that EFR makes use of the extensive form, while the present analysis is performed in the strategic form. The reason is that, by 'caution', a rational choice in the whole game implies a rational choice in all subgames that are not precluded from being reached by the player's own strategy.

To capture forward induction players must essentially deem any opponent strategy that is a rational choice infinitely more likely than any opponent strategy not having this property. An analysis incorporating this feature must involve a non-monotonic epistemic operator, which is called 'full belief' in the present analysis (cf. subsect. 3.4) and 'strong belief' by Battigalli & Siniscalchi ([11], Sect. 4). Here, 'full belief' is used only to define the event that the preferences of each player is 'fully admissibly consistent' with the preferences of his opponent. A standard monotonic epistemic operator ('certain belief') is, however, used for the interactive epistemology:

- each player certainly believes that the preferences of his opponent are fully admissibly consistent,
- each player certainly believes that his opponent certainly believes that he himself has preferences that are fully admissibly consistent, etc. ...

In contrast, Battigalli & Siniscalchi [11] use the non-monotonic operator 'strong belief' for the interactive epistemology, implying that an auxiliary operator (called 'correct strong belief') must be introduced for defining higher-order beliefs.

The fact that a non-monotonic epistemic operator is involved when capturing forward induction also means that the analysis must ensure that *all* rational choices for the opponent are included in the epistemic model. Battigalli & Siniscalchi [11] ensure this by employing a *complete* epistemic model, where all possible epistemic types for each player are represented. Instead, the present analysis achieves this by imposing that the preferences of the players are *characterized* by 'caution' and 'full belief of opponent rationality', meaning that the preferences are minimally complete (cf. subsect. 4.2). Since an ordinary monotonic operator is used for the interactive epistemology, there is no more need for a complete epistemic model here, than in usual epistemic analyses of rationalizability and permissibility.

Battigalli [9] has shown how EFR corresponds to the 'best rationalization principle'. This implies that some opponent strategies are neither completely rational nor completely irrational, but are considered to be at immediate degrees of rationality. Likewise, Stahl [38] provides an interpretation of IEWDS where strategies eliminated in the first round are completely irrational, while strategies eliminated in later rounds are at immediate degrees of rationality. The present analysis, in contrast, differentiates only between whether a strategy is maximal (i.e. a rational choice) or not. In particular, although a strategy that is weakly dominated on the set of all opponent strategies is a "stupid" choice, it need not be more "stupid" than any remaining admissible strategy, as this depends on the interactive analysis of the game. This point has been illustrated by the examples of Sect. 5.

Our paper has a predecessor in Samuelson [37], who also presents an epistemic analysis of admissibility that leads to a collection of sets for each player, called a 'generalized consistent pair'. Samuelson [37] requires that a player's choice set equals the set of strategies that are not weakly dominated on the union of choice sets that are deemed possible for the opponent; this corresponds to our requirement 'full belief of opponent rationality'. However, since 'caution' is not imposed, his analysis does not yield

$\{\{U, M\}\} \times \{\{L\}\}$ in our illustrative example (G_1). Furthermore, he imposes additional requirements (cf. his conditions (44) and (45)) that are incompatible with general existence. If each player is certain about the choice set of the opponent, one obtains a 'consistent pair' (cf. Börgers & Samuelson [19]), a concept that need not exist even when a generalized consistent pair exists. Ewerhart [29] modifies the concept of a consistent pair by adding 'caution'. However, since he removes minimal completeness to ensure general existence, his concept of a 'modified consistent pair' does not promote forward induction in G_2.

Note that 'caution' and 'full belief of opponent rationality' are requirements on the preferences (or beliefs) of players. Since minimal completeness is imposed by having preferences be characterized by 'caution' and 'full belief of opponent rationality', preferences need not be complete and cannot be represented by means of subjective probabilities (except through treating incomplete preferences as a *set* of complete preferences; cf. Aumann [7]). By not employing subjective probabilities, the analysis is related to the filter model of belief presented by Brandenburger [21]. By imposing requirements on the preferences of players rather than their choice, our paper follows a tradition in equilibrium analysis where concepts are characterized as equilibria in conjectures (cf. Blume, Brandenburger & Dekel [16]).[10]

6.2. **General Approach.** Through an approach where requirements are imposed on the consistency of each player's preferences with the preferences of his opponent – the 'consistent preferences' approach – we have characterized a fully permissible set as a choice set under common certain belief of full admissible consistency. We have thus provided an epistemic foundation for aspects of forward induction. The concept of full admissible consistency entails that types 'fully believe' that opponents choose rationally. The separation of 'full belief' of the rationality of opponent choice from 'certain belief' of the full admissible consistency of opponent types, which the 'consistent preferences' approach allows, means that the non-monotonic operator 'full belief' need not be used for the interactive epistemology.

The 'consistent preferences' approach has wider application; e.g., it can characterize rationalizability and permissibility as noted by Prop. 3 and Remark 1 of the present paper. Moreover, it is shown elsewhere how this approach may enhance our understanding of the epistemic conditions underlying backward induction by separating requirements on the assessment of opponent choice from 'certain belief' of the consistency of opponent types (cf. Asheim [4, 5]). Hence, the 'consistent preferences' approach is *not* an idiosyncratic approach exclusively designed for the characterization of the concept of fully permissible sets. Rather, it is an approach that follows naturally from the modeling of players as decision makers under uncertainty and which appears to have general interest for deductive game-theoretic analysis.

[10] In deductive game-theoretic analysis this — together with the requirement that preferences are minimally complete — is related to a property called 'coherence' by Gul [30].

Appendix A. The Decision-Theoretic Framework

The purpose of this appendix is to present the decision-theoretic terminology, notation and results utilized and referred to in the main text.

Consider a decision maker under uncertainty. Let F be a finite set of states, where the decision maker is uncertain about what state in F will be realized. Let Z be a finite set of outcomes. In the tradition of Anscombe & Aumann [1], the decision maker is endowed with a binary relation over all functions that to each element of F assigns an objective randomization on Z. Any such function $\mathbf{x}_F : F \to \Delta(Z)$ is called an *act* on F. Write $\mathbf{x}_F$ and $\mathbf{y}_F$ for acts on F. A *reflexive* and *transitive* binary relation on the set of acts on F is denoted by $\succeq_F$, where $\mathbf{x}_F \succeq_F \mathbf{y}_F$ means that $\mathbf{x}_F$ is *preferred* or *indifferent* to $\mathbf{y}_F$. As usual, let $\succ_F$ (*preferred to*) and $\sim_F$ (*indifferent to*) denote the asymmetric and symmetric parts of $\succeq_F$. A binary relation $\succeq_F$ on the set of acts on F is said to satisfy

- *objective independence* if $\mathbf{x}'_F \succ_F$ (respectively $\sim_F$) $\mathbf{x}''_F$ iff $\gamma\mathbf{x}'_F + (1-\gamma)\mathbf{y}_F \succ_F$ (respectively $\sim_F$) $\gamma\mathbf{x}''_F + (1-\gamma)\mathbf{y}_F$, whenever $0 < \gamma < 1$ and $\mathbf{y}_F$ is arbitrary.
- *nontriviality* if there exist $\mathbf{x}_F$ and $\mathbf{y}_F$ such that $\mathbf{x}_F \succ_F \mathbf{y}_F$.
- *continuity* if there exist $0 < \gamma < \delta < 1$ such that $\delta\mathbf{x}'_F + (1-\delta)\mathbf{x}''_F \succ_F \mathbf{y}_F \succ_F \gamma\mathbf{x}'_F + (1-\gamma)\mathbf{x}''_F$ whenever $\mathbf{x}'_F \succ_F \mathbf{y}_F \succ_F \mathbf{x}''_F$.

If $E \subseteq F$, let $\mathbf{x}_E$ denote the restriction of $\mathbf{x}_F$ to E. Define the *conditional* binary relation $\succeq_E$ by $\mathbf{x}'_F \succeq_E \mathbf{x}''_F$ if, for arbitrary $\mathbf{y}_F$, $(\mathbf{x}'_E, \mathbf{y}_{-E}) \succeq_F (\mathbf{x}''_E, \mathbf{y}_{-E})$, where $-E$ denotes $F\backslash E$. Say that the state $f \in F$ is *Savage-null* if $\mathbf{x}_F \sim_{\{f\}} \mathbf{y}_F$ for all acts $\mathbf{x}_F$ and $\mathbf{y}_F$ on F. A binary relation $\succeq_F$ is said to satisfy

- *conditional completeness* if, $\forall f \in F$, $\succeq_{\{f\}}$ is complete.
- *conditional continuity* if, $\forall f \in F$, there exist $0 < \gamma < \delta < 1$ such that $\delta\mathbf{x}'_F + (1-\delta)\mathbf{x}''_F \succ_{\{f\}} \mathbf{y}_F \succ_{\{f\}} \gamma\mathbf{x}'_F + (1-\gamma)\mathbf{x}''_F$ whenever $\mathbf{x}'_F \succ_{\{f\}} \mathbf{y}_F \succ_{\{f\}} \mathbf{x}''_F$.
- *non-null state independence* if $\mathbf{x}_F \succ_{\{e\}} \mathbf{y}_F$ iff $\mathbf{x}_F \succ_{\{f\}} \mathbf{y}_F$ whenever e and f are not Savage-null and $\mathbf{x}_F$ and $\mathbf{y}_F$ satisfy $\mathbf{x}_F(e) = \mathbf{x}_F(f)$ and $\mathbf{y}_F(e) = \mathbf{y}_F(f)$.

If $e, f \in F$ and $\succeq_F$ is conditionally complete, then e is deemed *infinitely more likely* than f ($e \gg f$) if e is not Savage-null and $\mathbf{x}_F \succ_{\{e\}} \mathbf{y}_F$ implies $(\mathbf{x}_{-\{f\}}, \mathbf{x}'_{\{f\}}) \succ_{\{e,f\}} (\mathbf{y}_{-\{f\}}, \mathbf{y}'_{\{f\}})$ for all $\mathbf{x}'_F$, $\mathbf{y}'_F$. According to this definition, f may, but need not, be Savage-null if $e \gg f$. Say that $\mathbf{y}_F$ is *maximal* w.r.t. $\succeq_E$ if there is no $\mathbf{x}_F$ such that $\mathbf{x}_F \succ_E \mathbf{y}_F$.

If $\upsilon : Z \to \mathbb{R}$ is a vNM utility function, abuse notation slightly by writing $\upsilon(x) = \sum_{z\in Z} x(z)\upsilon(z)$ whenever $x \in \Delta(Z)$ is an objective randomization. Say that $\mathbf{x}_E$ *strongly dominates* $\mathbf{y}_E$ w.r.t. υ if, $\forall f \in E$, $\upsilon(\mathbf{x}_E(f)) > \upsilon(\mathbf{y}_E(f))$. Say that $\mathbf{x}_E$ *weakly dominates* $\mathbf{y}_E$ w.r.t. υ if, $\forall f \in E$, $\upsilon(\mathbf{x}_E(f)) \geq \upsilon(\mathbf{y}_E(f))$, with strict inequality for some $e \in E$. Say that $\succeq_F$ is *admissible* on E ($\neq \emptyset$) if $\mathbf{x}_F \succ_F \mathbf{y}_F$ whenever $\mathbf{x}_E$ weakly dominates $\mathbf{y}_E$.

The following two representation results can now be stated. The first one — which follows directly from the von Neumann-Morgenstern theorem on expected utility representation — requires the notion of conditional representation: Say that $\succeq_F$ is *conditionally represented* by υ if (a) $\succeq_F$ is nontrivial and (b) $\mathbf{x}_F \succeq_{\{f\}} \mathbf{y}_F$ iff $\upsilon(\mathbf{x}_F(f)) \geq \upsilon(\mathbf{y}_F(f))$ whenever f is not Savage-null.

Proposition A1. *If $\succeq_F$ is reflexive and transitive, and satisfies objective independence, nontriviality, conditional completeness, conditional continuity, and non-null state independence, then there exists a vNM utility function $\upsilon : Z \to \mathbb{R}$ such that $\succeq_F$ is conditionally represented by υ.*

The second result, due to Blume, Brandenburger & Dekel ([15], Theorem 3.1), requires the notion of a *lexicographic probability system* (LPS) which consists of L levels of subjective probability distributions: If $L \geq 1$ and, $\forall \ell \in \{1, \ldots, L\}$, $\mu_\ell \in \Delta(F)$, then $\lambda = (\mu_1, ..., \mu_L)$ is an LPS on F. Let $\mathbf{L}\Delta(F)$ denote the set of LPSs on F, and let, for two utility vectors v and w, $v \geq_L w$ denote that, whenever $w_\ell > v_\ell$, there exists $\ell' < \ell$ such that $v_{\ell'} > w_{\ell'}$.

Proposition A2. *If $\succeq_F$ is complete and transitive, and satisfies objective independence, nontriviality, conditional continuity, and non-null state independence, then there exists a vNM utility function $\upsilon : Z \to \mathbb{R}$ and an LPS $\lambda = (\mu_1, ..., \mu_L) \in \mathbf{L}\Delta(F)$ such that $\mathbf{x}_F \succeq_F \mathbf{y}_F$ iff*

$$\left(\sum\nolimits_{f \in F} \mu_\ell(f) \upsilon(\mathbf{x}_F(f))\right)_{\ell=1}^{L} \geq_L \left(\sum\nolimits_{f \in F} \mu_\ell(f) \upsilon(\mathbf{y}_F(f))\right)_{\ell=1}^{L} .$$

If $F = F_1 \times F_2$ and $\succeq_F$ is a binary relation on the set of acts on F, then say that $\succeq_{F_1}$ is the *marginal* of $\succeq_F$ on F_1 if, $\mathbf{x}_{F_1} \succeq_{F_1} \mathbf{y}_{F_1}$ iff $\mathbf{x}_F \succeq_F \mathbf{y}_F$ whenever $\mathbf{x}_{F_1}(f_1) = \mathbf{x}_F(f_1, f_2)$ and $\mathbf{y}_{F_1}(f_1) = \mathbf{y}_F(f_1, f_2)$ for all (f_1, f_2).

Appendix B. Proofs

Proof of Proposition 1. Note that if $\emptyset \neq \Xi'_j \subseteq \Xi''_j \subseteq \Sigma_j$, then $\emptyset \neq \alpha_i(\Xi'_j) \subseteq \alpha_i(\Xi''_j) \subseteq \alpha_i(\Sigma_j)$. Repetitive use of this result implies that, $\forall g \geq 1$, $(\emptyset \neq)\ \Xi(g) \subseteq \Xi(g-1)\ (\subseteq \Sigma)$. From this monotonicity and the finiteness of Σ, it follows that $\Xi(g)$ converges in a finite number of iterations to Π with $(\emptyset \neq)\ \Pi = \alpha(\Pi)\ (\subseteq \Sigma)$. This establishes parts (i) and (ii). Let $\tilde{\Pi}$ denote the smallest rectangular collection that includes all Ξ satisfying $\Xi \subseteq \alpha(\Xi)$. There exists a collection Ξ satisfying $\Xi \subseteq \alpha(\Xi)$ since $\Pi \subseteq \alpha(\Pi)$; hence, $\Pi \subseteq \tilde{\Pi}$. If $\Xi \subseteq \alpha(\Xi)$, then $\Xi \subseteq \alpha(\Xi) \subseteq \alpha(\tilde{\Pi})$ since, $\forall i \in N$, α_i is monotone. Hence, $\tilde{\Pi} \subseteq \alpha(\tilde{\Pi})$ since $\alpha(\tilde{\Pi})$ is rectangular. As $(\emptyset \neq)\ \Pi \subseteq \tilde{\Pi} \subseteq \alpha(\tilde{\Pi}) \subseteq \alpha(\Xi(0)) = \Xi(1)\ (\subseteq \Sigma)$, repetitive use of the monotonicity result implies that, $\forall g \geq 1$, $(\emptyset \neq)\ \Pi \subseteq \tilde{\Pi} \subseteq \alpha(\tilde{\Pi}) \subseteq \alpha(\Xi(g-1)) = \Xi(g)\ (\subseteq \Sigma)$. Since $\Xi(g)$ converges to Π, it follows that $(\emptyset \neq)\ \Pi = \tilde{\Pi} = \alpha(\tilde{\Pi})\ (\subseteq \Sigma)$. This establishes part (iii). □

To prove Prop. 2, we first need to define the concept of permissible strategies. For any $(\emptyset \neq)\ X = X_1 \times X_2 \subseteq S$, write $\tilde{a}(X) := \tilde{a}_1(X_2) \times \tilde{a}_2(X_1)$, where

$$\tilde{a}_i(X_j) := S_i \setminus \{s_i \in S_i | \exists x_i \in \Delta(S_i) \text{ s.t. } x_i \text{ strongly dom. } s_i \text{ on } X_j \\ \text{or } x_i \text{ weakly dom. } s_i \text{ on } S_j\} .$$

Definition B1. Consider the sequence defined by $X(0) = S$ and, $\forall g \geq 1$, $X(g) = \tilde{a}(X(g-1))$. A pure strategy p_i is said to be *permissible* if $p_i \in \bigcap_{g=0}^{\infty} X_i(g)$.

Let $P = P_1 \times P_2$ denote the set of permissible strategy vectors. To characterize P, write for any $(\emptyset \neq)\ X = X_1 \times X_2 \subseteq S$, $a(X) := a_1(X_2) \times a_2(X_1)$, where

$$a_i(X_j) := \{p_i \in S_i | \exists (\emptyset \neq) Q_j \subseteq X_j \text{ s.t. } p_i \in S_i \backslash D_i(Q_j)\} .$$

Lemma B1. *For any $(\emptyset \neq)\ X_j \subseteq S_j$, $a_i(X_j) = \tilde{a}_i(X_j)$.*

Proof. Part 1: $a_i(X_j) \subseteq \tilde{a}_i(X_j)$. If $s_i \notin \tilde{a}_i(X_j)$, then $\exists x_i \in \Delta(S_i)$ s.t. x_i strongly dominates s_i on X_j or x_i weakly dominates s_i on S_j. From this it follows that $\forall\, (\emptyset \neq)\ Q_j \subseteq X_j$, $\exists x_i \in \Delta(S_i)$ s.t. x_i weakly dominates s_i on Q_j or S_j, implying that $\forall\, (\emptyset \neq)\ Q_j \subseteq X_j$, $s_i \in D_i(Q_j)$. This means that $s_i \notin a_i(X_j)$.

Part 2: $a_i(X_j) \supseteq \tilde{a}_i(X_j)$. If $s_i \in \tilde{a}_i(X_j)$, then there does not exist $x_i \in \Delta(S_i)$ s.t. x_i strongly dominates s_i on X_j or x_i weakly dominates s_i on S_j. Hence, by Pearce ([35], Lemmas 2 and 4), there exists an LPS $\lambda = (\mu_1, \mu_2) \in \mathbf{L}\Delta(S_j)$ with supp$\mu_1 \subseteq X_j$ and supp$\mu_2 = S_j$ such that s_i is maximal in $\Delta(S_i)$ w.r.t. the preferences represented by the vNM utility function u_i and the LPS λ (cf. Prop. A2). However, then there does not exist $x_i \in \Delta(S_i)$ s.t. x_i weakly dominates s_i on suppμ_1 ($\subseteq X_j$) or suppμ_2 ($= S_j$), implying that $s_i \notin D_i(Q_j)$ for $Q_j = \text{supp}\mu_1 \subseteq X_j$. This means that $s_i \in a_i(X_j)$. □

The following proposition is a straightforward implication of Lemma B1.

Proposition B1. *(i) The sequence defined by $X(0) = S$ and, $\forall g \geq 1$, $X(g) = a(X(g-1))$ converges to P in a finite number of iterations. (ii) $\forall i \in N$, $P_i \neq \emptyset$. (iii) $P = a(P)$. (iv) $\forall i \in N$, $p_i \in P_i$ if and only if there exists $X = X_1 \times X_2$ with $p_i \in X_i$ such that $X \subseteq a(X)$.*

Proof of Proposition 2. Using Prop. 1(ii), the definitions of $\alpha(\cdot)$ and $a(\cdot)$ imply that, $\forall i \in N$, $P_i^0 := \cup_{\sigma_i \in \Pi_i} \sigma_i = \cup_{\sigma_i \in \alpha_i(\Pi_i)} \sigma_i \subseteq a_i(P_i^0)$. Since $P^0 \subseteq a(P^0)$ implies $P^0 \subseteq P$ (by Prop. B1(iv)), it follows that, $\forall i \in N$, $\cup_{\sigma_i \in \Pi_i} \sigma_i \subseteq P_i$. □

For the proofs of Propositions 3 and 5, we first need to establish some properties of the operator 'certain belief' (cf. subsect. 3.3). Write $K^0 E := E$ and, for each $g \geq 1$, $K^g E := K K^{g-1} E$. Since $K_i(E \cap F) = K_i E \cap K_i F$, and since $K_i \emptyset = \emptyset$, conjunction, and positive and negative introspection imply that $K_i K_i E = K_i E$, it follows $\forall g \geq 2$, $K^g E = K_1 K^{g-1} E \cap K_2 K^{g-1} E \subseteq K_1 K_1 K^{g-2} E \cap K_2 K_2 K^{g-2} E = K_1 K^{g-2} E \cap K_2 K^{g-2} E = K^{g-1} E$. Even though the truth axiom ($K_i E \subseteq E$) is not satisfied, the present paper considers certain belief only of events (like $E := E_1 \cap E_2$ where, for each i, $E_i = \{\omega \in \Omega | t_i(\omega) \in T_i'\}$) that concerns the type vector. Mutual certain belief of any such event E implies that E is true: $KE = K_1 E \cap K_2 E \subseteq K_1 E_1 \cap K_2 E_2 = E_1 \cap E_2 = E$ since, for each i, $K_i E_i = E_i$. Hence, *(i)* $\forall g \geq 1$, $K^g E \subseteq K^{g-1} E$, and *(ii)* $\exists g' \geq 0$ such that $K^g E = CKE$ for $g \geq g'$ since Ω is finite, implying that $CKE = KCKE$.

Proof of Proposition 3. Part 1: If p_i is permissible, then there exists a belief system with $p_i \in C_i^{t_i(\omega)}$ for some $\omega \in CKA$. It is sufficient to show that one can construct a belief system with $A = \Omega = S \times T_1 \times T_2$ such that, $\forall i \in N$, $\forall p_i \in P_i$, there exists $t_i \in T_i$ with $p_i \in C_i^{t_i}$. Construct a belief system with, $\forall i \in N$, a one-to-one mapping $\mathbf{s}_i : T_i \to P_i$ from the set of types to the the set of permissible strategies. From Prop. B1(iii) it follows that, $\forall i \in N$, $\forall t_i \in T_i$, $\exists Q_j^{t_i} \subseteq P_i$ such that $\mathbf{s}_i(t_i) \in S_i \backslash D_i(Q_j^{t_i})$. Determine the set of opponent types that t_i does not deem Savage-null as follows: $T_j^{t_i} = \{t_j \in T_j | \mathbf{s}_j(t_j) \in Q_j^{t_i}\}$. Let $\succeq^{t_i}$ satisfy that $v_i^{t_i} \circ z = u_i$ and that $\mathbf{x} \succ^{t_i} \mathbf{y}$ iff $\mathbf{x}_{\beta_j}$ weakly dominates $\mathbf{y}_{\beta_j}$ for $\beta_j = \beta_j^{t_i} = \{(s_j, t_j) | s_j = \mathbf{s}_j(t_j) \text{ and } t_j \in T_j^{t_i}\}$ or $\beta_j = \kappa_j^{t_i} = S_j \times T_j^{t_i}$. This means that, $\forall i \in N$, $\forall t_i \in T_i$, $C_i^{t_i} = S_i \backslash D_i(Q_j^{t_i}) \ni \mathbf{s}_i(t_i)$ since $\mathbf{x}_{S_j} \succ_{S_j}^{t_i} \mathbf{y}_{S_j}$ iff $\mathbf{x}_{Q_j}$ weakly dominates $\mathbf{y}_{Q_j}$ for $Q_j = Q_j^{t_i}$ or $Q_j = S_j$, implying that, $\forall i \in N$, $\forall t_i \in T_i$, $\beta_j^{t_i} \subseteq [rat_j]_j$. Hence, $S \times T_1 \times T_2 = [u_1] \cap [cau_1] \cap B_1[rat_2] \cap [u_2] \cap [cau_2] \cap B_2[rat_1] = A$.

Part 2: If there exists a belief system with $p_i \in C_i^{t_i(\omega)}$ for some $\omega \in CKA$, then p_i is permissible. In view of Def. B1 and the properties preceding the proof, it is sufficient to show, $\forall g \geq 0$ and $\forall i \in N$, that $p_i \in X_i(g+1)$ if there exists a belief system with $p_i \in C_i^{t(\omega)}$ for some $\omega \in K^g A$. This is established by induction.

($g = 0$) Suppose $p_i \in S_i \backslash X_i(1) = S_i \backslash \tilde{a}_i(S_j)$. Then there exists $x_i \in \Delta(S_i)$ s.t. x_i weakly dominates s_i on S_j. Write $\mathbf{x}_{S_j}$ for the act on S_j that x_i can be viewed as, and write $\mathbf{y}_{S_j}$ for the act on S_j that s_i can be viewed as. Let $t_i = t_i(\omega)$ for some $\omega \in K^0 A = A$. Since $\omega \in A \subseteq [u_i] \cap [cau_i]$, it follows that $\mathbf{x}_{S_j} \succ^{t_i}_{S_j} \mathbf{y}_{S_j}$. Hence, there does not exist a belief system with $p_i \in C_i^{t_i(\omega)}$ for some $\omega \in K^0 A$ if $p_i \in S_i \backslash X_i(1)$. This means that $p_i \in X_i(1)$ if there exists a belief system with $p_i \in C_i^{t_i(\omega)}$ for some $\omega \in K^0 A$.

($g > 0$) Assume that it has been established, $\forall g' = 0, \dots, g-1$ and $\forall i \in N$, that $p_i \in X_i(g'+1)$ if there exists a belief system with $p_i \in C_i^{t(\omega)}$ for some $\omega \in K^{g'} A$. Suppose $p_i \in X_i(g) \backslash X_i(g+1) = X_i(g) \backslash \tilde{a}_i(X_j(g))$. Then there exists $x_i \in \Delta(S_i)$ s.t. x_i strongly dominates s_i on $X_j(g)$, which implies that, $\forall\ (\emptyset \neq)\ Q_j \subseteq X_j(g)$, x_i strongly dominates s_i on Q_j. Write $\mathbf{x}_{S_j}$ for the act on S_j that x_i can be viewed as, and write $\mathbf{y}_{S_j}$ for the act on S_j that s_i can be viewed as. Let $\mathbf{x}$ and $\mathbf{y}$ be acts on $S_j \times T_j$ that satisfy $\mathbf{x}(s_j, t_j) = \mathbf{x}_{S_j}(s_j)$ and $\mathbf{y}(s_j, t_j) = \mathbf{y}_{S_j}(s_j)$ for all (s_j, t_j). Then, $\forall\ (\emptyset \neq)\ Q_j \subseteq X_j(g)$, $\mathbf{x}_{Q_j \times T_j}$ weakly dominates $\mathbf{y}_{Q_j \times T_j}$. Let $t_i = t_i(\omega)$ for some $\omega \in K^g A$. Since $K^g A \subseteq K_i K^{g-1} A$ and $C_j^{t_j(\omega')} \subseteq X_j(g)$ whenever $\omega' \in K^{g-1} A$, it follows that, $\forall t_j \in T_j^{t_i}$, $C_j^{t_j} \subseteq X_j(g)$. Hence, $\forall\ (\emptyset \neq)\ \beta_j \subseteq [rat_j]_j \cap \kappa_j^{t_i}$, $\mathbf{x}_{\beta_j}$ weakly dominates $\mathbf{y}_{\beta_j}$, and, since $\omega \in [u_i] \cap B_i[rat_j]$, $\mathbf{x} \succ^{t_i} \mathbf{y}$ and $\mathbf{x}_{S_j} \succ^{t_i}_{S_j} \mathbf{y}_{S_j}$. Hence, there does not exist a belief system with $p_i \in C_i^{t_i(\omega)}$ for some $\omega \in K^g A$ if $p_i \in X_i(g) \backslash X_i(g+1)$. This means that $p_i \in X_i(g+1)$ if there exists a belief system with $p_i \in C_i^{t_i(\omega)}$ for some $\omega \in K^g A$. □

Proof of Proposition 4. It is sufficient to show that one can construct a belief system with $A^0 = \Omega = S \times \{t_1\} \times \{t_2\}$ such that, $\forall i \in N$, $\text{supp} x_i \subseteq C_i^{t_i}$, whenever (x_1, x_2) is a proper equilibrium. Let (x_1, x_2) be a proper equilibrium. By Blume, Brandenburger & Dekel's [16] Prop. 5, there exists a pair of preferences, $\succeq^{t_1}$ and $\succeq^{t_2}$, that are represented by $\upsilon_1^{t_1}$ and $\lambda^{t_1} = (\mu_1^{t_1}, \dots) \in \mathbf{L}\Delta(S_2 \times \{t_2\})$, and $\upsilon_2^{t_2}$ and $\lambda^{t_2} = (\mu_1^{t_2}, \dots) \in \mathbf{L}\Delta(S_1 \times \{t_1\})$, respectively — with $\upsilon_1^{t_1} \circ z = u_1$ and, $\forall s_2 \in S_2$, $\mu_1^{t_1}(s_2, t_2) = x_2(s_2)$, and $\upsilon_2^{t_2} \circ z = u_2$ and, $\forall s_1 \in S_1$, $\mu_1^{t_2}(s_1, t_1) = x_1(s_1)$ — satisfying, $\forall i \in N$, *(i)* $\text{supp} x_i \subseteq C_i^{t_i}$, *(ii)* $\kappa_j^{t_i} = S_j \times \{t_j\}$, and *(iii)* $(r_j, t_j) \gg (s_j, t_j)$ whenever $r_j \succ^{t_j} s_j$. Properties (ii) and (iii) imply that $\succeq^{t_i}$ is admissible on $C_j^{t_j} \times \{t_j\} = [rat_j]_j \cap \kappa_j^{t_i}$. By letting $\Omega = S \times \{t_1\} \times \{t_2\}$, it follows that $\Omega = [u_1] \cap [cau_1] \cap B_1^0[rat_2] \cap [u_2] \cap [cau_2] \cap B_2^0[rat_1] = A^0$. Hence, by property (i), $A^0 = \Omega = S \times \{t_1\} \times \{t_2\}$ and, $\forall i \in N$, $\text{supp} x_i \subseteq C_i^{t_i}$. □

Proof of Proposition 5. Part 1: If π_i is fully permissible, then there exists a belief system with $\pi_i = C_i^{t_i(\omega)}$ for some $\omega \in CK\bar{A}^0$. It is sufficient to show that one can construct a belief system with $\bar{A}^0 = \Omega = S \times T_1 \times T_2$ such that, $\forall i \in N$, $\forall \pi_i \in \Pi_i$, there exists $t_i \in T_i$ with $\pi_i = C_i^{t_i}$. Construct a belief system with, $\forall i \in N$, a one-to-one mapping $\boldsymbol{\sigma}_i : T_i \to \Pi_i$ from the set of types to the the set of fully permissible sets. From Prop. 1(ii) it follows that, $\forall i \in N$, $\forall t_i \in T_i$, $\exists \Psi_j^{t_i} \subseteq \Pi_i$ such that $\boldsymbol{\sigma}_i(t_i) = S_i \backslash D_i(Q_j^{t_i})$, where $Q_j^{t_i} := \{s_j \in S_j | \exists \sigma_j \in \Psi_j^{t_i} \text{ s.t. } s_j \in \sigma_j\}$. Determine the set of opponent types that t_i does not deem Savage-null as follows: $T_j^{t_i} = \{t_j \in T_j | \boldsymbol{\sigma}_j(t_j) \in \Psi_j^{t_i}\}$. Let $\succeq^{t_i}$ satisfy that $\upsilon_i^{t_i} \circ z = u_i$ and that $\mathbf{x} \succ^{t_i} \mathbf{y}$ iff $\mathbf{x}_{\beta_j}$ weakly dominates $\mathbf{y}_{\beta_j}$ for $\beta_j = \beta_j^{t_i} = \{(s_j, t_j) | s_j \in \boldsymbol{\sigma}_j(t_j) \text{ and } t_j \in T_j^{t_i}\}$ or $\beta_j = \kappa_j^{t_i} = S_j \times T_j^{t_i}$. This means that, $\forall i \in N$, $\forall t_i \in T_i$, $C_i^{t_i} = S_i \backslash D_i(Q_j^{t_i}) = \boldsymbol{\sigma}_i(t_i)$ since $\mathbf{x}_{S_j} \succ^{t_i}_{S_j} \mathbf{y}_{S_j}$ iff $\mathbf{x}_{Q_j}$ weakly dominates $\mathbf{y}_{Q_j}$ for $Q_j = Q_j^{t_i}$ or $Q_j = S_j$, implying that, $\forall i \in N$, $\forall t_i \in T_i$, $\beta_j^{t_i} = [rat_j]_j \cap \kappa_j^{t_i}$. Hence, $S \times T_1 \times T_2 = \bar{A}_1^0 \times \bar{A}_2^0 = \bar{A}^0$.

Part 2: If there exists a belief system with $\pi_i = C_i^{t_i(\omega)}$ for some $\omega \in CK\bar{A}^0$, then π_i is fully permissible. Consider any belief system for which $CK\bar{A}^0 \neq \emptyset$. Let, $\forall i \in N$, $T_i' := \{t_i(\omega) | \omega \in CK\bar{A}^0\}$ and $\Xi_i := \{C_i^{t_i} | t_i \in T_i'\}$. Note that, $\forall i \in N$ and $\forall t_i \in T_i'$, (s_j, t_j) is Savage-null acc. $\succeq^{t_i}$ if $t_j \in T_j \backslash T_j'$ since $CK\bar{A}^0 = KCK\bar{A}^0 \subseteq K_i CK\bar{A}^0$, implying that $T_j^{t_i} \subseteq T_j'$. Since, $\forall i \in N$ and $t_i \in T_i'$, $\mathbf{x} \succ^{t_i} \mathbf{y}$ iff $\mathbf{x}_{\beta_j}$ weakly dominates $\mathbf{x}_{\beta_j}$ for $\beta_j = \beta_j^{t_i} = [rat_j]_j \cap \kappa_j^{t_i}$ or $\beta_j = \kappa_j^{t_i} = S_j \times T_j^{t_i}$, it follows that $\mathbf{x}_{S_j} \succ_{S_j}^{t_i} \mathbf{y}_{S_j}$ iff $\mathbf{x}_{Q_j}$ weakly dominates $\mathbf{y}_{Q_j}$ for $Q_j = Q_j^{t_i}$ or $Q_j = S_j$, where $Q_j^{t_i} := \{s_j \in S_j | \exists \sigma_j \in \Psi_j^{t_i} \text{ s.t. } s_j \in \sigma_j\}$ and $\Psi_j^{t_i} := \{C_j^{t_j} | t_j \in T_j^{t_i}\} \subseteq \Xi_j$. This implies that, $\forall i \in N$ and $t_i \in T_i'$, $S_i \backslash D_i(Q_j^{t_i}) = C_i^{t_i}$, and $\Xi \subseteq \alpha(\Xi)$. Hence, by Prop. 1(iii), $\pi_i \in \Pi_i$ if there exists a belief system with $\pi_i = C_i^{t(\omega)}$ for some $\omega \in CK\bar{A}^0$. □

References

[1] Anscombe, F.J., Aumann, R.: A definition of subjective probability. Annals of Mathematical Statistics **34** (1963) 199–205.

[2] Armbruster, W., Böge, W.: Bayesian game theory. In "Game Theory and Related Topics" (Moeschlin, Pallaschke, Eds.), North-Holland, Amsterdam (1979).

[3] Asheim, G.B.: Defining rationalizability in 2-player extensive games. Memorandum No. 25, Dept. of Economics, University of Oslo (1994).

[4] Asheim, G.B.: On the epistemic foundation for backward induction. Memorandum No. 30, Dept. of Economics, University of Oslo (1999).

[5] Asheim, G.B.: Proper consistency. Memorandum No. 31, Dept. of Economics, University of Oslo (1999).

[6] Asheim, G.B., Dufwenberg, M.: Deductive reasoning in extensive games. University of Oslo (2000).

[7] Aumann, R.: Utility theory without the completeness axiom. Econometrica **30** (1962) 445–462.

[8] Battigalli, P.: Algorithmic solutions for extensive games. In "Decision Processes in Economic Proceedings" (Ricci, Ed.), Springer, Berlin (1989).

[9] Battigalli, P.: Strategic rationality orderings and the best rationalization principle. Games Econ. Beh. **13** (1996) 178–200.

[10] Battigalli, P.: On rationalizability in extensive games. J. Econ. Theory **74** (1997) 40–61.

[11] Battigalli, P., Siniscalchi, M.: Interactive beliefs and forward induction. Princeton University (2000).

[12] Ben-Porath, E.: Rationality, Nash equilibrium, and backwards induction in perfect information games. Rev. Econ. Stud. **64** (1997) 23–46.

[13] Ben-Porath, E., Dekel, E.: Coordination and the potential for self-sacrifice. J. Econ. Theory **57** (1992) 36–51.

[14] Bernheim, D.: Rationalizable strategic behavior. Econometrica **52** (1984) 1007–1028.

[15] Blume, L., Brandenburger, A., Dekel, E.: Lexicographic probabilities and choice under uncertainty. Econometrica **59** (1991) 61–79.

[16] Blume, L., Brandenburger, A., Dekel, E.: Lexicographic probabilities and equilibrium refinements. Econometrica **59** (1991) 81–98.

[17] Böge, W., Eisele, T.: On the solutions of Bayesian games. Int. J. Game Theory **8** (1979) 193–215.

[18] Börgers, T.: Weak dominance and approximate common knowledge. J. Econ. Theory **64** (1994) 265–276.

[19] Börgers, T., Samuelson, L.: 'Cautious' utility maximization and iterated weak dominance, Int. J. Game Theory **21** (1992) 13–25.

[20] Brandenburger, A.: Lexicographic probabilities and iterated admissibility. In "Economic Analysis of Markets and Games" (Dasgupta, Gale, Hart, Maskin, Eds.), MIT Press, Cambridge, MA (1992).
[21] Brandenburger, A.: On the existence of a 'complete' belief model. Harvard Business School Working Paper 99-056 (1998).
[22] Brandenburger, A., Dekel, E.: Hierarchies of beliefs and common knowledge. J. Econ. Theory **59** (1993) 189–198.
[23] Brandenburger, A., Keisler, H.J.: An impossibility theorem on beliefs in games. Harvard Business School Working Paper 00-010 (1999).
[24] Dekel, E., Fudenberg, D.: Rational behavior with payoff uncertainty. J. Econ. Theory **52** (1990) 243–267.
[25] Dekel, E., Gul, F.: Rationality and knowledge in game theory. In "Advances in Economics and Econometrics: Theory and Applications" (Kreps, Wallis, Eds.), Cambridge University Press, Cambridge, UK (1997).
[26] Dufwenberg, M.: Tie-break rationality and tie-break rationalizability. Working Paper 1994:29, Dept. of Economics, Uppsala University (1994).
[27] Epstein, L.G.: Preference, rationalizability and equilibrium. J. Econ. Theory **73** (1997) 1–29.
[28] Epstein, L.G., Wang, T.: "Beliefs about beliefs" without probabilities. Econometrica **64** (1996) 1343–1373.
[29] Ewerhart, C.: Rationality and the definition of consistent pairs. Int. J. Game Theory **27** (1998) 49–59.
[30] Gul, F.: Rationality and coherent theories of strategic behavior. J. Econ. Theory **70** (1997) 1–31.
[31] Hurkens, S.: Multi-sided pre-play communication by burning money. J. Econ. Theory **69** (1996) 186–197.
[32] Luce, D., Raiffa, H.: "Games and Decisions", Wiley, New York, NY (1957).
[33] Mertens, J.-M., Zamir, S.: Formulation of Bayesian analysis for games of incomplete information. Int. J. Game Theory **14** (1985) 1–29.
[34] Morris, S.: Alternative notions of belief. In "Epistemic Logic and the Theory of Decisions in Games" (Bacharach, Gerard-Varet, Mongin, Shin, Eds.), Kluwer, Dordrecht (1997).
[35] Pearce, D.G.: Rationalizable strategic behavior and the problem of perfection. Econometrica **52** (1984) 1029–1050.
[36] Rajan, U.: Trembles in the Bayesian foundations of solution concepts of games. J. Econ. Theory **82** (1998) 248–266.
[37] Samuelson, L.: Dominated strategies and common knowledge. Games Econ. Beh. **4** (1992) 284–313.
[38] Stahl, D.: Lexicographic rationality, common knowledge, and iterated admissibility. Econ. Letters **47** (1995) 155–159.
[39] Tan, T., Werlang, S.R.C.: The Bayesian foundations of solution concepts of games. J. Econ. Theory **45** (1988) 370–391.
[40] van Damme, E.: Stable equilibria and forward induction. J. Econ. Theory **48** (1989) 476–496.

Department of Economics, University of Oslo, P.O. Box 1095 Blindern, N-0317 Oslo, Norway; Department of Economics, Stockholm University, SE-10691 Stockholm, Sweden

E-mail address: g.b.asheim@econ.uio.no, md@ne.su.se